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Fivebrane instantons and Calabi-Yau fourfolds with flux

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Abstract: Using recent results on eleven-dimensional superspace, we compute the contribution of fivebrane instantons with four fermionic zeromodes in M-theory compactifications on Calabi-Yau fourfolds with flux. We find that no superpotential is generated in this case. This result is compatible with a certain flux-dependent modification of the arithmetic genus criterion.

Keywords: M-Theory, Flux compactifications, Superspaces.

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Contents

1. Introduction and summary

Recently, it has become clear that the problem of moduli stabilization may find its resolution in the context of flux compactifications (see e.g. [1 − 3] for recent reviews). In most recent models (starting with [4]) a crucial role is played by nonperturbative effects which can generate a superpotential for the Kähler moduli. Within the context of M-theory compactifications on Calabi-Yau fourfolds, as was first noted in [5], the nonperturbative effects arise from fivebrane instantons wrapping internal divisors. In a dual IIB picture this setup is equivalent to compactifications on Calabi-Yau threefolds, with instantons arising from D3-branes wrapping internal divisors.

In [5] Witten showed that in the absence of flux a necessary condition for the generation of a superpotential is that the divisor which the fivebrane wraps possesses a certain topological property: its arithmetic genus must be equal to one. When there are exactly two fermion zeromodes (corresponding to rigid isolated cycles) a superpotential is indeed generated. If more zeromodes are present, cancellations may occur. The lift of the arithmetic genus criterion to F-theory in general and IIB orientifolds in particular, was given by Robbins and Sethi in [6].

Recently attention has been drawn to the possibility that the arithmetic genus criterion may be violated in the presence of flux [7 − 9] (a discussion of the effects of flux was already presented in [6]). The authors of [9] defined a flux-dependent generalization of the arithmetic genus, χ_F, to be discussed in more detailed in the following. χ_F is not, strictly-speaking, an index: it cannot be defined as the dimension of the kernel minus the dimension of the cokernel of some operator. At present it is not clear what should the arithmetic genus criterion be replaced by in the presence of fluxes. In particular, it is not clear whether the arithmetic genus criterion should simply be replaced by the condition $\chi_F = 1$ or not. Moreover, it is conceivable that instantons with four or more fermionic zero-modes contribute to the superpotential,[1] as there exist higher-order fermionic terms in the worldvolume action of the fivebrane which may be used in order soak up the extra zero modes. Clarifying these issues is crucial for realistic model-building.

The computation of M-theory instantons goes back to the work of Becker et al [11]. These techniques were further elaborated by Harvey and Moore [12] in the context of G_2 compactifications. The subject of fivebrane instantons in M-theory has largely remained unexplored, mainly due to the exotic nature of the fivebrane worldvolume theory. Instanton effects in heterotic M-theory have been considered in [13 − 16].

Further progress beyond the computation of instantons with two zeromodes has been hindered by the lack of knowledge of the theta-expansions of the supervielbein and C-field in eleven-dimensional superspace. Recently there have been technical advances in this direction reported in [17], which applies the normal-coordinates approach [18] to the case of eleven-dimensional superspace. Using this method, the expression for linear backgrounds was derived to all orders in θ, i.e. up to and including terms of order θ^{32}. This constitutes significant progress, taking to account the fact that previously this expansion was known

[1] Instantons with more than two zeromodes are known to contribute to higher-derivative and/or multi-fermion couplings [10]. Here we examine whether such instantons can contribute to the *superpotential*.

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explicitly only to order θ^2 [19]. Results exact in the background fields were also presented up to and including terms of order θ^5.

It is the purpose of this paper to perform an explicit computation in the case of fivebrane instantons with four fermion zeromodes, in the context of M-theory compactifications on Calabi-Yau fourfolds in the presence of (normal) flux. We find that no superpotential is generated in this case. Therefore, our result does not rule out the possibility that in the presence of flux the arithmetic genus criterion should be replaced by the condition $\chi_F = 1$.

As this is a somewhat technical paper, in the following subsections of the introduction we have tried to put it in context and to summarize in a self-contained way the strategy and the result of the computation.

1.1 Review of the arithmetic genus criterion

In [5] Witten argued that M-theory compactifications on Calabi-Yau fourfolds may generate a nonzero superpotential in three dimensions through fivebrane instantons wrapping divisors of arithmetic genus one. We will now review his argument: consider a supersymmetric M-theory background of the form $\mathbb{R}^{1,2} \times X$, where X is a Calabi-Yau fourfold.[2] Provided a certain topological condition is satisfied, this is a consistent M-theory background [20, 21]. Compactification on X results in an $\mathcal{N} = 2$ theory in three dimensions (four real supercharges). This theory is very similar to a supersymmetric $\mathcal{N} = 1$ theory in four dimensions, and we may think of it (although this is not necessary) as a dimensional reduction from four to three dimensions. Similarly to the case in four dimensions, the kinetic terms are obtained by integration over the whole superspace, whereas the Yukawa couplings and the mass terms are obtained by integrating over half the superspace (F-terms). Crucially, powerful nonrenormalization theorems prevent radiative corrections to the F-terms.

Let us now describe the structure of the so-called 'linear multiplets', which play a distinguished role in the discussion of [5] and in the following: the bosonic part of a linear multiplet in four dimensions consists of a second-rank antisymmetric tensor and a real scalar. The fact that the antisymmetric tensor is dual in four dimensions to a scalar, can be promoted at the level of superfields to a duality between linear and chiral supermultiplets. Upon reduction to three dimensions the chiral multiplets give rise to chiral multiplets, whereas the linear multiplets become vector multiplets. In analogy to the situation in four dimensions, a vector in three dimensions is dual to the a scalar *provided there is no Chern-Simons term* arising from the compactification on the fourfold. In absence of fluxes there is indeed no Chern-Simons term which could obstruct the dualization, but this is generally no longer the case in the presence of fluxes [23, 24].

To be more explicit: upon compactification of M-theory on a Calabi-Yau fourfold, one obtains b_2 vectors from the threeform gauge field

$$C = \sum_{I=1}^{b_2} A^I(x) \wedge \omega_I + \dots , \tag{1.1}$$

[2] Eventually we will work in Euclideanized eleven-dimensional space.

where x is a (three-dimensional) spacetime coordinate and $\{\omega_I,\ I=1,\dots b_2\}$ is a basis of $H^2(X,\mathbb{R})$, which of course coincides with $H^{1,1}(X,\mathbb{R})$ for a Calabi-Yau fourfold. In the absence of a Chern-Simons term in three dimensions the A^Is can be dualized to b_2 scalars, which we will call the 'dual scalars' ϕ_D^I, $d\phi_D^I = \star dA^I$. Note that perturbatively there are Peccei-Quinn symmetries whereby the dual scalars are shifted by constants; as we will see in the following, these continuous symmetries can be broken by instantons to discrete subgroups thereof. In addition to the ϕ_D^Is there are b_2 scalars, ϕ^I, from the deformations of the Kähler form J,

$$J = \sum_{I=1}^{b_2} \phi^I(x)\omega_I\ . \tag{1.2}$$

After dualization, the bosonic fields of each vector multiplet in three dimensions (these are the 'descendants' of the linear multiplets in four dimensions) consist of a pair of real scalars $(\phi^I,\ \phi_D^I)$. The superpotential W depends holomorphically on $\phi^I + i\phi_D^I$.

Following [5], we note that all terms in the superpotential depend on the vector multiplets. Indeed if there were any terms in the superpotential which did not depend on the vector multiplets, they could be computed by scaling up the metric of X (since such terms would be independent of the Kähler class, which belongs to the vector multiplets). But in the limit where the metric is scaled up, M-theory reduces to supergravity and $\mathbb{R}^{1,2}\times X$ becomes an exact solution — showing that there is no superpotential in this case.

To look for instantons which may generate a superpotential, we note that the threeform gauge field is (magnetically) sourced by the fivebrane. Hence, a relevant instanton in three dimensions is seen from the eleven-dimensional point-of-view as a fivebrane wrapping a six-cycle Σ in the Calabi-Yau fourfold. In order for the instanton to preserve half the supersymmetry (so that it may generate an F-term), the cycle Σ must be a holomorphic divisor. This fact is re-derived in detail in section 4.2, in the presence of normal flux.

As can be verified explicitly, the contribution of the instanton includes the classical factor

$$\int d^2\theta_0\ e^{-(\mathrm{Vol}_\Sigma + i\phi_D)}\,, \tag{1.3}$$

where Vol_Σ is the volume (in units of the eleven-dimensional Planck length l_P) of the six-cycle the fivebrane is wrapping, and ϕ_D is the linear combination of dual scalars which constitutes the superpartner of Vol_Σ. I.e. the scalars $(\mathrm{Vol}_\Sigma,\ \phi_D)$ form the real and imaginary parts of a chiral superfield, as is expected from the holomorphic property of the superpotential (which is, in its turn, a consequence of supersymmetry). For the generation of a superpotential, the fermionic terms in the fivebrane action should conspire so as to soak up all but two of the fermion zeromodes. The Grassmann integration in (1.3) above is the integration over the remaining fermionic zeromodes. As was then argued in [5], apart from the classical factor above, the superpotential should be independent of the Kähler class. This is because the dependence on ϕ_D is fixed by the magnetic charge of the instanton, and so the dependence on Vol_Σ is in its turn fixed by holomorphy.

Apart from the classical factor above, the steepest-slope approximation of the path integral around the fivebrane instanton includes a one-loop determinant, which is independent of the Kähler class but depends holomorphically on the complex structure moduli. The one-loop result is in fact exact, as higher loops do not contribute to the superpotential. This can be seen as follows: higher loops would be proportional to positive powers of l_P and would therefore scale as inverse powers of the volume; but, as already mentioned, apart from the classical factor the superpotential cannot depend on the Kähler class.

A necessary criterion for a divisor Σ to contribute to the superpotential is that its arithmetic genus χ,

$$\chi = \sum_{p=0}^{3} (-1)^p h^{p,0}(\Sigma) \,, \tag{1.4}$$

is equal to one. This was arrived at in [5] by the following line of arguments: first note that, in the limit where Σ is scaled up, the $\mathrm{U}(1)$ rotations along the normal direction to Σ inside the fourfold become an exact symmetry (dubbed 'W-symmetry' in [5]) of M-theory. On the other hand, in the absence of fluxes the worldvolume theory of the fivebrane has a one-loop W-anomaly equal to χ. It must then be that the exponential in (1.3) has W-charge equal to $-\chi$.[3] Moreover, it is straightforward to see that the fermionic zeromode measure carries W-charge equal to one. It follows that a necessary condition for the generation of a superpotential is $\chi = 1$; this is the arithmetic genus criterion.

1.2 Caveats to the arithmetic genus criterion

As already anticipated in [5], the arithmetic genus criterion may be violated in cases where the assumption of W-symmetry fails. This can occur if there are couplings of the fermions to normal derivatives of the background fields (i.e. normal to the divisor Σ inside X). Indeed, in the presence of flux such couplings are present already in the 'minimal' quadratic-fermion action $\theta \not{D} \theta$, where $\not{D}$ is a flux-dependent Dirac operator which we will define more precisely in the following. Even in the absence of flux, W-violating couplings will generally be present at higher orders in the fermions, they will however be suppressed in the large-volume limit.

A further complication is the following: in the presence of flux, there is a Chern-Simons term in the three-dimensional low-energy supergravity,

$$T_{IJ} d\phi^I \wedge A^J \,, \tag{1.5}$$

which will a priori obstruct the straightforward dualization of the vectors A^I to scalars ϕ_D^I [23, 24]. One may therefore worry about the fate of holomorphy, on which the derivation of the arithmetic genus criterion relied. (Recall that the holomorphic property of the superpotential allowed us to take the large-volume limit in which the W-symmetry becomes exact). The object T_{IJ} which enters the Chern-Simons term above is a constant symmetric

[3] Note that Witten's paper [5] was written before the cancellation of the normal-bundle anomaly of the fivebrane was properly understood in [22]. It would be interesting to derive this result directly using the techniques of [22].

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matrix given by

$$T_{IJ} := \frac{\partial^2 T}{\partial\phi^I \partial\phi^J} = \int_X F \wedge \omega_I \wedge \omega_J$$
$$T := \frac{1}{2}\int_X F \wedge J \wedge J = \frac{1}{2} T_{IJ}\phi^I\phi^J \, , \tag{1.6}$$

where F is the internal component of the fourform flux. Its quantization condition is equivalent to the expansion

$$F = \sum_{a=1}^{b_4} n^a \omega_a + \sum_{I=1}^{b_2} dA^I \wedge \omega_I \, , \tag{1.7}$$

where $\{\omega_a, \ a = 1 \dots b_4\}$ is a basis of $H^4(X, \mathbb{Z})$, and the n^as are integers. An additional effect of the flux is the gauging of the the Peccei-Quinn isometries. The gauging is completely determined by the constant matrix T_{IJ}.

Contrary perhaps to the naive expectation, the dualization of vectors to scalars can proceed more-or-less straightforwardly also in the case with fluxes. Let us assume for simplicity that we work in a basis of $H^2(X, \mathbb{R})$ such that T_{IJ} is diagonal, and for the moment let's assume that the complex structure moduli are frozen. It then follows from the work of [25] (which is based on general results on three-dimensional gauged supergravities [26]) that *(i)* the isometries $\phi_D^I \to \phi_D^I +$ constant corresponding to zero eigenvalues of T_{IJ} are *not* gauged and *(ii)* if $\phi_D^I \to \phi_D^I +$ constant is an isometry which *does* get gauged, the superpotential cannot depend on ϕ_D (nor can it depend on the Kähler modulus ϕ^I, by holomorphy).[4] This picture is consistent with the conclusions of [29] who find (in the context of IIA string theory) that those isometries which are gauged by the flux are protected from quantum corrections.

1.3 The results of the present paper

In the presence of fluxes, the scalar potential of the low-energy three-dimensional supergravity is still given in terms of the holomorphic superpotential W, but in addition will also generally depend on T. On the other hand the fermion bilinears

$$\chi^I \chi^J D_I D_J W + \text{c.c.} \, , \tag{1.8}$$

where D_I is a Kähler-covariant derivative, solely depend on the holomorphic superpotential, W, even in the presence of fluxes [26]. (Fermion mass terms of the form $\bar{\chi}^I \chi^J M_{IJ}$ do depend on T, as we will see in section 5.1). Hence, a straightforward way to obtain instanton corrections to the superpotential is to compute the coupling (1.8).

For the purpose of examining the possible generation of a superpotential by instanton effects, it follows from the discussion in section 1.2 that we only need examine whether the coupling (1.8) is generated for fermions χ^I which correspond to zero eigenvalues of T_{IJ}

[4]On the other hand, if there are additional fields which are charged under the gauge potential, this conclusion may be relaxed [27]. We thank M. Haack for pointing this out. In the present context, such phenomena may arise presumably in the presence of M2 branes [28] and will not be examined here.

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(we may consider a basis where T_{IJ} is diagonal, for simplicity). Hence, we may assume that the Kähler moduli corresponding to nonzero eigenvalues of T_{IJ} are frozen to zero.[5] In other words we can assume, as follows from (1.2), (1.6), that we are in the region of the Kähler moduli space where:

$$\int_X F \wedge J \wedge \omega_I = 0; \qquad I = 1 \dots b_2 \,. \tag{1.9}$$

If no such region exists, i.e. if T_{IJ} has no zero eigenvalues, all isometries are gauged and there can be no superpotential dependence on the Kähler moduli: the superpotential is protected against instanton contributions. Moreover, condition (1.9) implies that

$$\omega_I \lrcorner F = 0 \,, \tag{1.10}$$

for all ω_Is corresponding to zero eigenvalues of T_{IJ}. This observation simplifies somewhat the rather tedious computational task of this paper. In particular, we may assume we are in the region of the Kähler moduli space where the flux is primitive: $J \lrcorner F = 0$. Furthermore, for the purposes of the present computation we may assume that the complex structure moduli are frozen to values such that the internal fourform flux is of type (2,2). These are exactly the conditions which ensure that *the flux is compatible with supersymmetry*, as we will see in detail in section 4.1.

Despite the fact that certain conceptual subtleties remain, there are clear rules for instanton computations in M-theory first put forward in [11] and subsequently elucidated in [12]. We will schematically describe the procedure here, relegating the details to the main body of the paper. In order to compute the instanton contribution to the coupling (1.8), one first decomposes the eleven-dimensional gravitino in terms of three-dimensional fermions χ^I,

$$\Psi_m = \chi^I \otimes \Omega_{I,m} \xi \,, \tag{1.11}$$

where ξ is the covariantly constant spinor of the Calabi-Yau fourfold[6] and Ω_I is a one-form on X valued in the Clifford algebra $Cl(TX)$. Next, from the fivebrane action one reads off the coupling of the eleven-dimensional gravitino to the fivebrane worldvolume fermion θ, schematically:

$$V = \sum_n c_n \Psi \theta^{2n+1} \,, \tag{1.12}$$

for some, possibly flux-dependent, 'coefficients' c_n. The coupling V is the 'gravitino vertex operator'. Finally, to read off the coefficient $D_I D_J W$ in (1.8) one evaluates the correlator $\langle VV \rangle$ in the worldvolume theory of the fivebrane.

[5] Since the superpotential does not depend on these moduli, we may fix them to any desired value: the superpotential, and hence the coupling (1.8), is independent of the chosen value. The present choice leads to convenient computational simplifications. Examples of fourfolds for which there are choices of fourform flux such that T_{IJ} vanishes identically, were examined in [30].

[6] In the presence of flux, the internal space becomes a warped Calabi-Yau. As we will see, however, the effect of the warp factor can be ignored at leading order in the large-volume expansion.

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Note that the worldvolume fermions are valued in the normal bundle to the fivebrane, which is the sum of $T\mathbb{R}^3$ (after passing to Euclidean signature) and the normal bundle to the divisor inside the fourfold. Thus, each worldvolume fermion should be thought of as tensored with a two-component spinor of Spin(3). The main result of the present paper is that *instantons with exactly four fermionic zeromodes do not contribute to the superpotential.* In deriving this result we have made the simplifying assumption that both the curvature of the worldvolume self-dual tensor as well as the pull-back of the threeform flux onto the worldvolume vanish. This is what we call the condition of 'normal flux'.

One major technical difficulty with the present computation is the explicit expansion of the fivebrane action in terms of the worldvolume fermion, the so-called 'theta-expansion'. This, in its turn, stems from the theta-expansion of the eleven-dimensional background superfields on which the fivebrane action depends. Until recently, this expansion had only been fully worked out to quadratic order in the fermions. The present computation is now possible thanks to the recent results of [17] in which, among other things, the theta-expansion of the eleven-dimensional superfields was computed explicitly to fifth order in the fermions.

We should at this point elaborate on what we mean by 'the fivebrane action'. The fivebrane dynamics was given in terms of covariant field equations in [31, 32]. For the application we are interested in, however, one needs to work with an action. As is well known, the worldvolume theory of the fivebrane contains a self dual antisymmetric tensor which renders the formulation of an action problematic. A covariant supersymmetric action for the fivebrane can be constructed with the help of an auxiliary scalar [33]. Alternatively, the auxiliary field can be eliminated at the expense of explicitly breaking Lorentz invariance [34]. The equivalence of all different formulations was shown in [35]. Here we will use the covariant action of [33].

An important cautionary remark is in order. In [36] Witten pointed out that a useful way to define the action of a self-dual field is in terms of a Chern-Simons theory in one dimension higher. This definition, for spacetime dimensions higher than two, involves a suitable generalization of the notion of spin structure — on a choice of which the self-dual action depends. These issues have been recently clarified by Belov and Moore [37, 38]. Unfortunately, the action of [33] does not take these topological aspects into account; it is however at present our only available (covariant) supersymmetric action for the fivebrane.

1.4 Outline

We now give a detailed plan of the rest of the paper. Section 2 relies on [17] treating the theta-expansion of the various superfields of the eleven-dimensional background, with the aim of applying it to the worldvolume theory of the fivebrane. The theta expansion of the sixform potential was not considered in [17], and this is addressed in section 2.2. The worldvolume theory of the fivebrane is considered in section 3 in the framework of the covariant action of [33]. Eventually we make the simplifying assumption that the flux is 'normal', i.e. that both the field-strength of the worldvolume antisymmetric tensor and the pull-back of the background threeform flux onto the fivebrane worldvolume, vanish. The

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main result of this section is the form of the gravitino vertex operator in the case of normal flux, equation (3.5).

Section 4.1 considers M-theory backgrounds of the form of a warp product $\mathbb{R}^{1,2} \times_w X$, where X is a Calabi-Yau fourfold. (Eventually we Wick-rotate to Euclidean signature and take the large-volume limit in which the warp factor becomes trivial). Requiring $\mathcal{N} = 2$ supersymmetry in three dimensions (four real supercharges) implies certain restrictions on the fourform flux, equation (4.4). Next we consider fivebrane instantons such that the worldvolume wraps a six-cycle $\Sigma \subset X$ and we assume that X can be thought of as the total space of the normal bundle of Σ inside X. As discussed in the introduction, this approximation becomes more accurate as the size of Σ is scaled up. Imposing the normal-flux condition, the form of the background flux simplifies further, equations (4.7), (4.8).

In section 4.2 we show that, in the case of normal flux, demanding that the instanton preserve one-half the supersymmetry of the background implies that Σ is an (anti)holomorphic cycle. Section 4.3 treats the worldvolume fermion zeromodes of the flux-dependent Dirac operator, equation (4.19). After decomposing the background fermion in terms of forms on the fivebrane, we derive the explicit expression of the fermion zeromodes (4.36). This result agrees with the analysis of [8, 9], in the case of normal flux and provided the warp factor is trivial. This can be consistently taken to be the case in the large-volume limit, as explained in section 4.1.

In section 5 we finally come to the main subject of the paper, the instanton contributions to the superpotential. Section 5.1 discusses the Kaluza-Klein Ansatz for the gravitino, equation (5.1). Next, the Kaluza-Klein ansätze for the gravitino as well as for the fermion zeromodes are substituted into the expression (3.5) for the gravitino vertex operator. The result of the fermion zeromode integration in the case of two zeromodes is briefly discussed in section 5.2. In section 5.3 it is shown that in the case of four fermion zeromodes the result of the zeromode integration is zero. I.e. in this case the instanton contribution to the superpotential vanishes.

The appendices contain several useful technical details. For quick reference, we have also included an index of our conventions and notation in section D.

2. Theta-expansions

This section examines the theta-expansions of the various eleven-dimensional superfields. Except for the expansion of the sixform which is given in section 2.2, these were treated in reference [17] to which the reader is referred for further details. For reasons which are explained below (3.5), for our purposes we will not need the explicit form of the Ψ^2 contact terms. It also suffices to keep terms up to and including order θ^3. Also note that we are using standard superembedding notation, whereby target-space indices are underlined. Further explanation of the notation can be found in appendix D.

2.1 Vielbein and threeform

Using the formulæ in [17], to which the interested reader is referred for further details, we find

$$
\begin{aligned}
E_m{}^{\underline{a}} = e_m{}^{\underline{a}} &- \frac{i}{2}(\mathcal{D}_m\theta\Gamma^{\underline{a}}\theta) + \frac{1}{24}(\mathcal{D}_m\theta\mathfrak{G}\Gamma^{\underline{a}}\theta) + \frac{1}{24}(\theta\mathcal{R}_{\underline{np}}\mathcal{I}_m{}^{\underline{np}}\Gamma^{\underline{a}}\theta) \\
&- i(\Psi_m\Gamma^{\underline{a}}\theta) + \frac{1}{6}(\Psi_m\mathfrak{G}\Gamma^{\underline{a}}\theta) + \frac{1}{6}(\Psi_{\underline{np}}\mathcal{I}_m{}^{\underline{np}}\Gamma^{\underline{a}}\theta) + \mathcal{O}(\Psi^2, \theta^5)\,, \qquad (2.1)
\end{aligned}
$$

where

$$
\begin{aligned}
(\mathfrak{G})_{\underline{\alpha}}{}^{\underline{\beta}} &:= \frac{1}{576}\Big\{(\theta\Gamma^{\underline{abcdef}})_{\underline{\alpha}}(\theta\Gamma_{\underline{ef}})^{\underline{\beta}} - 2(\theta\Gamma_{\underline{e}})_{\underline{\alpha}}(\theta\Gamma^{\underline{abcde}})^{\underline{\beta}} - 16(\theta\Gamma^{\underline{a}})_{\underline{\alpha}}(\theta\Gamma^{\underline{bcd}})^{\underline{\beta}} \\
&\qquad +24(\theta\Gamma^{\underline{ab}})_{\underline{\alpha}}(\theta\Gamma^{\underline{cd}})^{\underline{\beta}}\Big\}G_{\underline{abcd}}\,, \qquad (2.2) \\
(\mathcal{I}_m{}^{\underline{ef}})_{\underline{\alpha}}{}^{\underline{\beta}} &:= -\frac{1}{48}\Big\{(\theta\Gamma_{\underline{ab}})_{\underline{\alpha}}(\theta\Gamma_m{}^{\underline{abef}})^{\underline{\beta}} + 4(\theta\Gamma_{m\underline{a}})_{\underline{\alpha}}(\theta\Gamma^{\underline{aef}})^{\underline{\beta}} - 4(\theta\Gamma_{\underline{ab}})_{\underline{\alpha}}(\theta\Gamma^{\underline{abe}})^{\underline{\beta}}e_m{}^{\underline{f}} \\
&\qquad +6(\theta\Gamma_m)_{\underline{\alpha}}(\theta\Gamma^{\underline{ef}})^{\underline{\beta}} - 12(\theta\Gamma_{\underline{a}})_{\underline{\alpha}}(\theta\Gamma^{\underline{ae}})^{\underline{\beta}}e_m{}^{\underline{f}}\Big\}\;. \qquad (2.3)
\end{aligned}
$$

Using (2.1) we find for the Green-Schwarz metric

$$
\begin{aligned}
g_{mn} = G_{mn} &- \frac{1}{4}(\mathcal{D}_m\theta\Gamma^{\underline{a}}\theta)(\mathcal{D}_n\theta\Gamma_{\underline{a}}\theta) - i(\mathcal{D}_{(m}\theta\Gamma_{n)}\theta) + \frac{1}{12}(\mathcal{D}_{(m}\theta\mathfrak{G}\Gamma_{n)}\theta) \\
&+ \frac{1}{12}(\theta\mathcal{R}_{\underline{pq}}\mathcal{I}_{(m}{}^{\underline{pq}}\Gamma_{n)}\theta) - 2i(\Psi_{(m}\Gamma_{n)}\theta) + \frac{1}{3}(\Psi_{(m}\mathfrak{G}\Gamma_{n)}\theta) \\
&+ \frac{1}{3}(\Psi_{\underline{pq}}\mathcal{I}_{(m}{}^{\underline{pq}}\Gamma_{n)}\theta) - (\Psi_{(m}\Gamma^{\underline{a}}\theta)(\mathcal{D}_{n)}\theta\Gamma_{\underline{a}}\theta) + \mathcal{O}(\Psi^2, \theta^5)\;. \qquad (2.4)
\end{aligned}
$$

Similarly, for the pull-back of the three-form we find

$$
\begin{aligned}
C_{mnp} = c_{mnp} &- \frac{3i}{2}(\mathcal{D}_{[m}\theta\Gamma_{np]}\theta) + \frac{1}{8}(\mathcal{D}_{[m}\theta\mathfrak{G}\Gamma_{np]}\theta) + \frac{1}{8}(\theta\mathcal{R}_{\underline{pq}}\mathcal{I}_{[m}{}^{\underline{pq}}\Gamma_{np]}\theta) \qquad (2.5) \\
&- \frac{3}{4}(\mathcal{D}_{[m}\theta\Gamma_n{}^{\underline{a}}\theta)(\mathcal{D}_{p]}\theta\Gamma_{\underline{a}}\theta) - 3i(\Psi_{[m}\Gamma_{np]}\theta) - (\Psi_{[m}\Gamma_n{}^{\underline{a}}\theta)(\mathcal{D}_{p]}\theta\Gamma_{\underline{a}}\theta) \\
&- 2(\Psi_{[m}\Gamma^{\underline{a}}\theta)(\mathcal{D}_n\theta\Gamma_{p]\underline{a}}\theta) + \frac{1}{2}(\Psi_{[m}\mathfrak{G}\Gamma_{np]}\theta) + \frac{1}{2}(\Psi_{\underline{nq}}\mathcal{I}_{[m}{}^{\underline{nq}}\Gamma_{np]}\theta) + \mathcal{O}(\Psi^2, \theta^5)\;.
\end{aligned}
$$

2.2 Sixform

The θ-expansion for C_6 was not given in [17], but the same methods can be applied in this case. First we note that the C_6-field satisfies

$$
7\partial_{[\underline{M}_1}C_{\underline{M}_2\ldots\underline{M}_7\}} = G_{\underline{M}_1\ldots\underline{M}_7}. \qquad (2.6)
$$

Up to a gauge choice, the following is a solution of the Bianchi identity (2.6) at each order in the θ expansion:

$$
\begin{gathered}
C^{(0)}_{\underline{\mu}_1\ldots\underline{\mu}_6} = C^{(0)}_{\underline{\mu}_1\ldots\underline{\mu}_5\underline{m}_1} = \ldots C^{(0)}_{\underline{\mu}_1\underline{m}_1\ldots\underline{m}_5} = 0\,, \\
7\partial_{[\underline{m}_1}C^{(0)}_{\underline{m}_2\ldots\underline{m}_7]} = G^{(0)}_{\underline{m}_1\ldots\underline{m}_7} \qquad (2.7)
\end{gathered}
$$

and

$$
\begin{aligned}
C^{(n+1)}_{\underline{\mu}_1\cdots\underline{\mu}_6} &= \frac{1}{n+7}\ \theta^{\underline{\lambda}} G^{(n)}_{\underline{\lambda}\underline{\mu}_1\cdots\underline{\mu}_6} \\
C^{(n+1)}_{\underline{\mu}_1\cdots\underline{\mu}_5\underline{m}_1} &= \frac{1}{n+6}\ \theta^{\underline{\lambda}} G^{(n)}_{\underline{\lambda}\underline{\mu}_1\cdots\underline{\mu}_5\underline{m}_1} \\
C^{(n+1)}_{\underline{\mu}_1\cdots\underline{\mu}_4\underline{m}_1\underline{m}_2} &= \frac{1}{n+5}\ \theta^{\underline{\lambda}} G^{(n)}_{\underline{\lambda}\underline{\mu}_1\cdots\underline{\mu}_4\underline{m}_1\underline{m}_2} \\
C^{(n+1)}_{\underline{\mu}_1\underline{\mu}_2\underline{\mu}_3\underline{m}_1\underline{m}_2\underline{m}_3} &= \frac{1}{n+4}\ \theta^{\underline{\lambda}} G^{(n)}_{\underline{\lambda}\underline{\mu}_1\underline{\mu}_2\underline{\mu}_3\underline{m}_1\underline{m}_2\underline{m}_3} \\
C^{(n+1)}_{\underline{\mu}_1\underline{\mu}_2\underline{m}_1\cdots\underline{m}_4} &= \frac{1}{n+3}\ \theta^{\underline{\lambda}} G^{(n)}_{\underline{\lambda}\underline{\mu}_1\underline{\mu}_2\underline{m}_1\cdots\underline{m}_4} \\
C^{(n+1)}_{\underline{\mu}\underline{m}_1\cdots\underline{m}_5} &= \frac{1}{n+2}\ \theta^{\underline{\lambda}} G^{(n)}_{\underline{\lambda}\underline{\mu}\underline{m}_1\cdots\underline{m}_5} \\
C^{(n+1)}_{\underline{m}_1\cdots\underline{m}_6} &= \frac{1}{n+1}\ \theta^{\underline{\lambda}} G^{(n)}_{\underline{\lambda}\underline{m}_1\cdots\underline{m}_6}\,, \qquad n \geq 0\,.
\end{aligned}
\tag{2.8}
$$

Using the fact that

$$
G_{\underline{a}_1\cdots\underline{a}_5\underline{\alpha}_1\underline{\alpha}_2} = -i(\Gamma_{\underline{a}_1\cdots\underline{a}_5})_{\underline{\alpha}_1\underline{\alpha}_2}\,, \tag{2.9}
$$

we find for the right-hand sides of the equations (2.8),

$$
\begin{aligned}
\theta^{\underline{\lambda}} G_{\underline{\lambda}\underline{\mu}_1\cdots\underline{\mu}_6} &= 6iE_{(\underline{\mu}_1}{}^{\underline{a}_1}\ldots E_{\underline{\mu}_5}{}^{\underline{a}_5}E_{\underline{\mu}_6)}{}^{\underline{\alpha}}(\Gamma_{\underline{a}_1\cdots\underline{a}_5}\theta)_{\underline{\alpha}} \\
\theta^{\underline{\lambda}} G_{\underline{\lambda}\underline{\mu}_1\cdots\underline{\mu}_5\underline{m}} &= -\,5iE_{\underline{m}}{}^{\underline{a}_1}E_{(\underline{\mu}_1}{}^{\underline{a}_2}\ldots E_{\underline{\mu}_4}{}^{\underline{a}_5}E_{\underline{\mu}_5)}{}^{\underline{\alpha}}(\Gamma_{\underline{a}_1\cdots\underline{a}_5}\theta)_{\underline{\alpha}} \\
&\quad + iE_{\underline{\mu}_1}{}^{\underline{a}_1}\ldots E_{\underline{\mu}_5}{}^{\underline{a}_5}E_{\underline{m}}{}^{\underline{\alpha}}(\Gamma_{\underline{a}_1\cdots\underline{a}_5}\theta)_{\underline{\alpha}} \\
\theta^{\underline{\lambda}} G_{\underline{\lambda}\underline{\mu}_1\cdots\underline{\mu}_4\underline{m}_1\underline{m}_2} &= 4iE_{\underline{m}_1}{}^{\underline{a}_1}E_{\underline{m}_2}{}^{\underline{a}_2}E_{(\underline{\mu}_1}{}^{\underline{a}_3}E_{\underline{\mu}_2}{}^{\underline{a}_4}E_{\underline{\mu}_3}{}^{\underline{a}_5}E_{\underline{\mu}_4)}{}^{\underline{\alpha}}(\Gamma_{\underline{a}_1\cdots\underline{a}_5}\theta)_{\underline{\alpha}} \\
&\quad + 2iE_{\underline{\mu}_1}{}^{\underline{a}_1}\ldots E_{\underline{\mu}_4}{}^{\underline{a}_4}E_{[\underline{m}_1}{}^{\underline{a}_5}E_{\underline{m}_2]}{}^{\underline{\alpha}}(\Gamma_{\underline{a}_1\cdots\underline{a}_5}\theta)_{\underline{\alpha}} \\
\theta^{\underline{\lambda}} G_{\underline{\lambda}\underline{\mu}_1\underline{\mu}_2\underline{\mu}_3\underline{m}_1\underline{m}_2\underline{m}_3} &= -\,3iE_{\underline{m}_1}{}^{\underline{a}_1}E_{\underline{m}_2}{}^{\underline{a}_2}E_{\underline{m}_3}{}^{\underline{a}_3}E_{(\underline{\mu}_1}{}^{\underline{a}_4}E_{\underline{\mu}_2}{}^{\underline{a}_5}E_{\underline{\mu}_3)}{}^{\underline{\alpha}}(\Gamma_{\underline{a}_1\cdots\underline{a}_5}\theta)_{\underline{\alpha}} \\
&\quad + 3iE_{\underline{\mu}_1}{}^{\underline{a}_1}E_{\underline{\mu}_2}{}^{\underline{a}_2}E_{\underline{\mu}_3}{}^{\underline{a}_3}E_{[\underline{m}_1}{}^{\underline{a}_4}E_{\underline{m}_2}{}^{\underline{a}_5}E_{\underline{m}_3]}{}^{\underline{\alpha}}(\Gamma_{\underline{a}_1\cdots\underline{a}_5}\theta)_{\underline{\alpha}} \\
\theta^{\underline{\lambda}} G_{\underline{\lambda}\underline{\mu}_1\underline{\mu}_2\underline{m}_1\cdots\underline{m}_4} &= +\,2iE_{\underline{m}_1}{}^{\underline{a}_1}\ldots E_{\underline{m}_4}{}^{\underline{a}_4}E_{(\underline{\mu}_1}{}^{\underline{a}_5}E_{\underline{\mu}_2)}{}^{\underline{\alpha}}(\Gamma_{\underline{a}_1\cdots\underline{a}_5}\theta)_{\underline{\alpha}} \\
&\quad + 4iE_{\underline{\mu}_1}{}^{\underline{a}_1}E_{\underline{\mu}_2}{}^{\underline{a}_2}E_{[\underline{m}_1}{}^{\underline{a}_3}E_{\underline{m}_2}{}^{\underline{a}_4}E_{\underline{m}_3}{}^{\underline{a}_5}E_{\underline{m}_4]}{}^{\underline{\alpha}}(\Gamma_{\underline{a}_1\cdots\underline{a}_5}\theta)_{\underline{\alpha}} \\
\theta^{\underline{\lambda}} G_{\underline{\lambda}\underline{\mu}\underline{m}_1\cdots\underline{m}_5} &= -\,iE_{\underline{m}_1}{}^{\underline{a}_1}\ldots E_{\underline{m}_5}{}^{\underline{a}_5}E_{\underline{\mu}}{}^{\underline{\alpha}}(\Gamma_{\underline{a}_1\cdots\underline{a}_5}\theta)_{\underline{\alpha}} \\
&\quad + 5iE_{\underline{\mu}}{}^{\underline{a}_1}E_{[\underline{m}_1}{}^{\underline{a}_2}\ldots E_{\underline{m}_4}{}^{\underline{a}_5}E_{\underline{m}_5]}{}^{\underline{\alpha}}(\Gamma_{\underline{a}_1\cdots\underline{a}_5}\theta)_{\underline{\alpha}} \\
\theta^{\underline{\lambda}} G_{\underline{\lambda}\underline{m}_1\cdots\underline{m}_6} &= 6iE_{[\underline{m}_1}{}^{\underline{a}_1}\ldots E_{\underline{m}_5}{}^{\underline{a}_5}E_{\underline{m}_6]}{}^{\underline{\alpha}}(\Gamma_{\underline{a}_1\cdots\underline{a}_5}\theta)_{\underline{\alpha}}\,.
\end{aligned}
\tag{2.10}
$$

In the following we will only need the part ΔC_6 of C_6 which is linear in the gravitino. Plugging the expressions for the vielbein components given in [17] into (2.10) we obtain

$$
\begin{aligned}
\Delta C_{m_1\ldots m_6} &= -\,6i(\Psi_{[m_1}\Gamma_{m_2\ldots m_6]}\theta) + 10(\Psi_{[m_1}\Gamma^{\underline{a}}\theta)(\mathcal{D}_{m_2}\theta\Gamma_{m_3\ldots m_6]\underline{a}}\theta) \\
&\quad + (\Psi_{[m_1}\mathfrak{G}\Gamma_{m_2\ldots m_6]}\theta) + (\Psi_{\underline{pq}}\mathcal{I}_{[m_1}{}^{\underline{pq}}\Gamma_{m_2\ldots m_6]}\theta) \\
&\quad - 5(\Psi_{[m_1}\Gamma_{m_2\ldots m_5\underline{a}}\theta)(\mathcal{D}_{m_6]}\theta\Gamma^{\underline{a}}\theta) + \mathcal{O}(\Psi^2,\theta^5)\,.
\end{aligned}
\tag{2.11}
$$

3. Fivebrane action

We are now ready to consider the application of the theta-expansion discussed in the previous section to the case of the fivebrane worldvolume action. As already mentioned

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in the introduction, we will adopt the covariant framework of [33] to which the reader is referred for more details. The main result of this section is the gravitino vertex operator, equation (3.5) below. To improve the presentation, we have relegated the details of the derivation to appendix C.

The fivebrane action is of the form

$$S = S_1 + S_2 + S_3 \,, \tag{3.1}$$

where

$$\begin{aligned} S_1 &:= T_{M5} \int_\Sigma d^6x \sqrt{-det(g_{mn} + i\widetilde{H}_{mn})} \\ S_2 &:= T_{M5} \int_\Sigma d^6x \sqrt{-g}\, \frac{1}{4} \widetilde{H}_{mn} H^{mn} \\ S_3 &:= T_{M5} \int_\Sigma \Big(C_6 + \frac{1}{2} F_3 \wedge C_3 \Big) \end{aligned} \tag{3.2}$$

and $T_{M5} \sim l_P^{-6}$ is the fivebrane tension. Moreover, we have made the following definitions

$$\begin{aligned} H_{mnp} &:= F_{mnp} - C_{mnp} \\ H_{mn} &:= H_{mnp} v^p \\ \widetilde{H}^{mn} &:= \frac{1}{6\sqrt{-g}} \epsilon^{mnpqrs} v_p H_{qrs} \\ v_p &:= \frac{\partial_p a}{\sqrt{-g^{mn}\partial_m a \partial_n a}} \,, \end{aligned} \tag{3.3}$$

where F_{mnp} is the field-strength of the world-volume chiral two-form and a is an auxiliary world-volume scalar. It follows from the above definitions that

$$det(\delta_m{}^n + i\widetilde{H}_m{}^n) = 1 + \frac{1}{2} tr\widetilde{H}^2 + \frac{1}{8}(tr\widetilde{H}^2)^2 - \frac{1}{4} tr\widetilde{H}^4 \ . \tag{3.4}$$

3.1 The gravitino vertex operator

In the case of normal flux, i.e. when the world-volume two-form tensor is flat ($F_{mnp} = 0$) and the pull-back of the three-form potential onto the fivebrane vanishes ($c_{mnp} = 0$), the expression for the gravitino vertex operator simplifies considerably. Skipping the details of the derivation, which can be found in appendix C, the final result reads:

$$\boxed{\begin{aligned} V = T_{M5} \int_\Sigma d^6x \sqrt{-G}\, \Big\{ & 2(\Psi_m \Gamma^m \theta) + i(\Psi_{\underline{m}} V^{(2)\underline{m}}) + \frac{i}{3}(\Psi_m \mathfrak{G} \Gamma^m \theta) \\ & + \frac{i}{3}(\Psi_{\underline{pq}} \mathcal{I}_m{}^{\underline{pq}} \Gamma^m \theta) + \mathcal{O}(\Psi^2, \theta^5) \Big\} \ . \end{aligned}} \tag{3.5}$$

We can now see why the Ψ^2 contact-terms can be neglected. As is easy to verify, Ψ^2 terms first appear in the θ-expansion at order θ^4. Consequently, a single vertex-operator insertion V_{Ψ^2} is needed to saturate the four fermion zeromodes –which is the case examined here. A single insertion, however, is proportional to T_{M5} and is of order $\mathcal{O}(l_P^6)$ relative to two vertex-operator insertions: the latter give a contribution proportional to T_{M5}^2. Clearly, this analysis is valid provided the 'radius' of the six-cycle is much larger than the Planck length, $\mathrm{Vol}_\Sigma >> l_P^6$.

As was shown in the case of the M-theory membrane [39] and is also expected in the case of the fivebrane [40], the first higher-order correction to the world-volume action occurs at order l_P^4. Hence it would be inconsistent to include contact terms without considering the higher-order derivative corrections to the world-volume action. Moreover, at order l_P^6 (eight derivatives) there are higher-order curvature corrections to the background supergravity action[7] which, as was explained in [17], modify the θ-expansion of all superfields.

3.2 Quadratic fermion terms

It follows from the preceding sections that in a bosonic background ($\Psi_{\underline{m}}^{\underline{\alpha}} = 0$) the part of the Lagrangian quadratic in θ (this is the analogue of equations (38), (39) of [42]) is given by

$$\begin{aligned}
\mathcal{L}^{(quad)} =& \frac{i}{2}\sqrt{det(A_i{}^j)}(A^{-1})^{(mn)}(\theta\Gamma_m\mathcal{D}_n\theta) \\
&- \frac{\epsilon^{lpqrs}{}_m}{6\sqrt{-G}}\sqrt{det(A_i{}^j)}(A^{-1})^{[mn]}(\theta\Gamma_{(n}\mathcal{D}_{l)}\theta)a_p(F_{qrs}-c_{qrs}) \\
&- \frac{\epsilon^{klpqrs}}{24\sqrt{-G}}\sqrt{det(A_i{}^j)}(A^{-1})_{kl}a_p \\
&\quad\times\Big\{(F_{qrs}-c_{qrs})\Big[a^m a^n(\theta\Gamma_m\mathcal{D}_n\theta)+(\theta\Gamma^m\mathcal{D}_m\theta)\Big]+3(\theta\Gamma_{qr}\mathcal{D}_s\theta)\Big\} \\
&- \frac{i\epsilon^{klpqrs}}{24\sqrt{-G}}a_k a^m(F_{lpq}-c_{lpq}) \\
&\quad\times\Big\{(F_{rst}-c_{rst})\Big[a^t a^n(\theta\Gamma_n\mathcal{D}_m\theta)+\frac{1}{2}(\theta\Gamma^t\mathcal{D}_m\theta)\Big]+\frac{1}{2}(\theta\Gamma_{rs}\mathcal{D}_m\theta)\Big\} \\
&- \frac{i\epsilon^{klpqrs}}{48\sqrt{-G}}a_k a^n(F_{lpq}-c_{lpq})(F_{rs}{}^t-c_{rs}{}^t)(\theta\Gamma_n\mathcal{D}_t\theta) \\
&- \frac{i\epsilon^{klpqrs}}{2\times 5!\sqrt{-G}}\Big\{15a^t a_k(F_{lpt}-c_{lpt})(\theta\Gamma_{qr}\mathcal{D}_s\theta)-10a^t a_k(F_{lpq}-c_{lpq})(\theta\Gamma_{rt}\mathcal{D}_s\theta) \\
&\qquad -5F_{klp}(\theta\Gamma_{qr}\mathcal{D}_s\theta)-(\theta\Gamma_{klpqr}\mathcal{D}_s\theta)\Big\}\,.
\end{aligned} \tag{3.6}$$

Note that $\mathcal{L}^{(quad)}$ is related to $V^{(1)\underline{m}}_{\ \underline{\alpha}}$ in a simple way.

[7]The eleven-dimensional supergravity admits a supersymmetric deformation at order l_P^3 (five derivatives) [41]. On a topologically-nontrivial spacetime M such that $p_1(M) \neq 0$, this deformation can be removed by a C-field redefinition, at the cost of shifting the quantization condition of the fourform field-strentgh.

Normal flux. In this case the part of the Lagrangian quadratic in the fermions simplifies to

$$\mathcal{L}^{(quad)} = \frac{i}{2}\Big\{(\theta\Gamma^m\mathcal{D}_m\theta) + \frac{\epsilon^{klpqrs}}{5!\sqrt{-G}}(\theta\Gamma_{klpqr}\mathcal{D}_s\theta)\Big\} \ . \tag{3.7}$$

After Wick-rotating we obtain

$$\mathcal{L}^{(quad)} = -(\theta\Gamma^m\mathcal{D}_m\theta)\,, \tag{3.8}$$

where we have taken (C.23) into account, and we have noted that after gauge-fixing the physical fermion modes satisfy $P^+\theta = \theta$.

4. Supersymmetric cycles

This section is devoted to the analysis of the conditions for a supersymmetric six-cycle, and the derivation of the worldvolume fermionic zeromodes in the presence of (normal) flux.

4.1 M-theory on fourfolds

We start by reviewing M-theory on a Calabi-Yau fourfold with flux. Let the eleven-dimensional metric be of the form

$$ds^2 = \Delta^{-1}ds_3^2 + \Delta^{1/2}ds_8^2\,, \tag{4.1}$$

where ds_3^2 is the metric of three-dimensional Minkowski space, Δ is a warp factor, and ds_8^2 is the metric on X. Let us also decompose the eleven-dimensional Majorana-Weyl supersymmetry parameter η in terms of a real anticommuting spinor ϵ along the three-dimensional Minkowski space, and a real chiral spinor ξ on X:

$$\eta = \Delta^{-1/4}\epsilon\otimes\xi \ . \tag{4.2}$$

As was first shown in [43], the requirement of $\mathcal{N}=1$ supersymmetry in three dimensions (two real supercharges) leads to the condition

$$\nabla_m\xi = 0\,, \tag{4.3}$$

i.e. the 'internal' spinor is covariantly constant with respect to the connection associated with the metric g_{mn} on X. Under the Ansatz (4.2), requiring $\mathcal{N}=2$ supersymmetry in three dimensions implies the existence of two real covariantly-constant spinors $\xi_{1,2}$ of the same chirality. It follows that X is a Calabi-Yau four-fold. In the following we shall combine $\xi_{1,2}$ into a complex chiral spinor on X, $\xi := \xi_1 + i\xi_2$. An antiholomorphic $(0,4)$ fourform Ω and a complex structure J on X can be constructed as bilinears of ξ, as is discussed in detail in appendix B. Moreover, supersymmetry imposes the following conditions on the components of the fourform field-strength:

$$G = \mathrm{Vol}_3\wedge d\Delta^{-3/2} + F\,, \tag{4.4}$$

where F is a fourform on X which is purely $(2,2)$ and traceless, $J \lrcorner F = 0$, with respect to the complex structure J on X. We have denoted by Vol_3 the volume element of the three-dimensional Minkowski space. Finally, the warp factor is constrained by the Bianchi identities to satisfy

$$d \star d \, \log\Delta = \frac{1}{3} F \wedge F - \frac{2}{3} (2\pi)^4 \beta X_8 \,, \tag{4.5}$$

where β is a constant of order l_P^6, and the Hodge star is with respect to the metric on X. The second term on the right-hand side of the equation above is a higher-order correction related to the fivebrane anomaly. In general there will be other corrections of the same order which should also be taken into account. However, it can be argued that in the large-radius approximation it is consistent to only take the above correction into account (see [44], for example).

In the large-volume limit $g^{CY} = t g_0^{CY} + \ldots$, $t \to \infty$, the two terms on the right-hand side of (4.5) scale like t^{-3} relative to the left-hand side and can be neglected. It is therefore consistent to take the warp factor to be trivial, $\Delta = 1$ [45]. We will henceforth assume this to be the case. In particular, it follows from (4.4) that the fourform's only nonzero components are along the Calabi-Yau fourfold. Note that the integrated version of equation (4.5),

$$\int_X F \wedge F + \frac{\beta}{12} \, \chi(X) = 0 \,, \tag{4.6}$$

is the tadpole cancellation condition.

Finally, note that the normal flux condition, together with the constraints of supersymmetry on the fourform flux explained in section 4.1, imply that F is of the form

$$F_{mnpq} = 4 \widetilde{F}_{[mnp} K_{q]} + 4 \widetilde{F}^*_{[mnp} K^*_{q]} \,, \tag{4.7}$$

where $\widetilde{F}$ obeys

$$J \lrcorner \widetilde{F} = 0; \qquad \iota_K \widetilde{F} = \iota_{K^*} \widetilde{F} = 0 \tag{4.8}$$

and K is a complex vector field normal to the fivebrane worldvolume, see eq. (4.20) below.

The above results can be extended to include more general fluxes [46, 47]. In this case the internal manifold generally ceases to be Calabi-Yau.

4.2 Supersymmetric cycles

Consider a bosonic superembedding of the fivebrane ($X^{\underline{m}}(\sigma)$, $\theta^{\underline{\mu}}(\sigma) = 0$) in a bosonic background ($\Psi_{\underline{m}}{}^{\underline{\alpha}} = 0$), where σ^m is the coordinate on the fivebrane worldvolume. The fivebrane action is invariant under superdiffeomorphisms

$$\delta_\zeta Z^{\underline{M}} = \zeta^{\underline{A}} E_{\underline{A}}{}^{\underline{M}} \tag{4.9}$$

such that

$$\mathcal{L}_\zeta E_{\underline{M}}{}^{\underline{A}} = -(\partial_{\underline{M}} + \Omega_{\underline{MB}}{}^{\underline{A}}) \zeta^{\underline{B}} - \zeta^{\underline{B}} T_{\underline{BM}}{}^{\underline{A}} = 0 \ . \tag{4.10}$$

This can be seen by first noting that

$$\mathcal{L}_\zeta C_3 = d(\iota_\zeta C_3) + \iota_\zeta G_4 \ . \tag{4.11}$$

The first term on the right-hand side pulls back to a total derivative on the fivebrane worldvolume, which can be compensated by a gauge transformation. The pull-back of the second term on the right-hand side vanishes for a bosonic background at $\theta = 0$, as can be seen by (4.13) below and by taking into account that the only nonzero components of G_4 are $G_{\underline{ab\alpha\beta}}$ and $G_{\underline{abcd}}$. Similarly, the WZ term transforms under (4.9) as

$$\int_{W_6} \mathcal{L}_\zeta (C_6 + \frac{1}{2} F_3 \wedge C_3) = \int_{W_6} \iota_\zeta (G_7 + \frac{1}{2} H_3 \wedge G_4) \,, \tag{4.12}$$

where we have dropped a total derivative from the integrand. Again, this vanishes for a bosonic background at $\theta = 0$. Finally, the Green-Schwarz metric is manifestly invariant under (4.9), (4.10).

Condition (4.10) can be solved for ζ, order by order in a θ-expansion. By taking the torsion constraints into account, it can be shown that

$$\begin{aligned} \zeta^{\underline{\alpha}} &= \eta^{\underline{\alpha}}(X) + \mathcal{O}(\theta^2) \\ \zeta^{\underline{a}} &= i(\eta \Gamma^{\underline{a}} \theta) + \mathcal{O}(\theta^3) \,, \end{aligned} \tag{4.13}$$

where $\eta^{\underline{\alpha}}$ is a Killing spinor,

$$\mathcal{D}_{\underline{m}} \eta^{\underline{\alpha}}(X) = 0 \ . \tag{4.14}$$

Transformation (4.9) corresponds to a zero mode iff it can be compensated by a κ-transformation, i.e. iff there exists $\kappa^{\underline{\alpha}}(\sigma)$ such that

$$\eta^{\underline{\alpha}}(X(\sigma)) + \kappa^{\underline{\alpha}}(\sigma) = 0 \ . \tag{4.15}$$

On the other hand κ satisfies $\kappa^{\underline{\beta}} \bar{\Gamma}_{\underline{\beta}}{}^{\underline{\alpha}} = \kappa^{\underline{\alpha}}$, where

$$\begin{aligned} \bar{\Gamma}(\sigma) := \frac{1}{\sqrt{det(\delta_r{}^s + i\widetilde{H}_r{}^s)}} \Big\{ & \frac{1}{6!} \frac{\epsilon^{m_1 \dots m_6}}{\sqrt{-g}} \Gamma_{m_1 \dots m_6} + \frac{i}{2} \Gamma_{mnp} \widetilde{H}^{mn} v^p \\ & - \frac{1}{16} \frac{\epsilon^{m_1 \dots m_6}}{\sqrt{-g}} \widetilde{H}_{m_1 m_2} \widetilde{H}_{m_3 m_4} \Gamma_{m_5 m_6} \Big\} \,, \end{aligned} \tag{4.16}$$

so that $\bar{\Gamma}^2 = 1$. Hence (4.15) is equivalent to

$$\eta^{\underline{\beta}}(X(\sigma))(1 - \bar{\Gamma}(\sigma))_{\underline{\beta}}{}^{\underline{\alpha}} = 0 \,, \tag{4.17}$$

with $\bar{\Gamma}(\sigma)$ evaluated for the bosonic fivebrane superembedding in the bosonic background. To summarize: the 'global' zero modes are given by

$$\theta^{\underline{\alpha}}(\sigma) = \eta^{\underline{\alpha}}(X(\sigma)) \,, \tag{4.18}$$

where η satisfies (4.14), (4.17). Consequently, $\theta^{\underline{\alpha}}$ is annihilated by $\mathcal{D}_m = \partial_m X^{\underline{m}} \mathcal{D}_{\underline{m}}$ and hence obeys the Dirac equation on the fivebrane:

$$\Gamma^m \mathcal{D}_m \theta = 0 \,, \tag{4.19}$$

which follows from the quadratic part of the fivebrane action (3.8). I.e. 'global' zero modes give rise to zero modes on the fivebrane. The converse is not generally true.

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Supersymmetric cycles in the case of normal flux. For a large six-cycle Σ, X can be approximated by the total space of the normal bundle of Σ in X as in [5]. Equivalently, Σ can be specified by a complex vector field K on X such that

$$ds^2(X) = G_{mn} d\sigma^m \otimes d\sigma^n + K \otimes K^* \,, \tag{4.20}$$

where $G_{mn}(\sigma)$ is the metric of Σ, and $K^m G_{mn} = 0$. We shall normalize K as in appendix B, $|K|^2 = 2$, in which case the determinants of the metrics on X, Σ are equal.

The kappa-symmetry projector simplifies considerably in the case of normal flux. Passing to the static gauge and Wick-rotating, condition (4.17) can be seen to be equivalent to

$$\Big(1 - \frac{K^m K^{*n} \epsilon_{mn}{}^{m_1 \dots m_6}}{2 \times 6! \sqrt{G}} \Gamma_{m_1 \dots m_6}\Big)\xi = 0 \,. \tag{4.21}$$

Furthermore, using the formulæ in the appendix, equation (4.21) can be rewritten as

$$P^+ \xi = \xi; \qquad P^+ := \frac{1}{2}\Big(1 + \frac{1}{2} K^m K^{*n} \Gamma_{mn} \Gamma_9\Big) \,. \tag{4.22}$$

The normal vector K is not a priori holomorphic with respect to the complex structure of X. However, it is straightforward to see from (4.22) that

$$J_m{}^n K_n = -i K_m \,. \tag{4.23}$$

It follows that in the case of normal flux, supersymmetric cycles are antiholomorphic cycles.

4.3 Zero modes

We are now ready to come to the analysis of the fermionic zeromodes on the worldvolume of the fivebrane. The main result of this section is given in (4.34) below. In the process we make contact with the earlier results of [8, 9]. The form of the Dirac operator in the linear approximation was derived in [48].

A note on notation: in the remainder of the paper, lower-case Latin letters from the middle of the alphabet $(m, n, \dots)$ denote indices along X (as opposed to indices along the fivebrane worldvolume).

Spinors-forms correspondence on X. Using formulæ (B.6) in appendix B we can see that any chiral spinor λ_+ on X can be expanded as

$$\lambda_+ = \Phi^{(0,0)}\xi + \Phi^{(2,0)}_{mn}\gamma^{mn}\xi + \Phi^{(4,0)}_{mnpq}\gamma^{mnpq}\xi \,, \tag{4.24}$$

where $\Phi^{(p,0)}$ is a $(p,0)$-form with respect to the complex structure J. I.e. $\Phi^{(2,0)}$ is in the **6** of SU(4) and $\Phi^{(4,0)}$ is a singlet. Similarly in the case of an antichiral spinor λ_- we can expand

$$\lambda_- = \Phi^{(1,0)}_m \gamma^m \xi + \Phi^{(3,0)}_{mnp}\gamma^{mnp}\xi \,, \tag{4.25}$$

where $\Phi^{(1,0)}$ is in the **4** of SU(4) and $\Phi^{(3,0)}$ is in the $\bar{\mathbf{4}}$. More succinctly, the equations above are nothing but the equivalence

$$\begin{aligned} S_+ &\cong \Lambda^{(\text{even},0)} \\ S_- &\cong \Lambda^{(\text{odd},0)} \,, \end{aligned} \tag{4.26}$$

which can be shown to hold in the case of a Calabi-Yau manifold.

Spinors-forms correspondence on the fivebrane. We will now assume that the fivebrane wraps a supersymmetric cycle, as described above. Ignoring the three flat directions for simplicity, after gauge-fixing the kappa-symmetry the fermions on the worldvolume of the fivebrane transform as sections of the tensor product

$$\begin{aligned} S_+ \otimes (S_+(N) \oplus S_-(N)) &\cong \Lambda^{(0,0)} \oplus \Lambda^{(2,0)} \oplus K \oplus (K \otimes \Lambda^{(2,0)}) \\ &\cong \Lambda^{(0,0)} \oplus \Lambda^{(2,0)} \oplus \Lambda^{(0,1)} \oplus \Lambda^{(0,3)} \, , \end{aligned} \tag{4.27}$$

where $S_\pm(N)$ are the positive-, negative-chirality spin bundles associated to the normal bundle N of Σ in X, $\Lambda^{(p,0)}$ is the bundle of $(p,0)$-forms on Σ, and K is the canonical bundle of Σ. The first equivalence above can be shown by taking the adjunction formula into account, and the triviality of the canonical bundle of X. The second equivalence is proven by noting that $K \otimes \Lambda^{(3-p,0)} \cong \Lambda^{(0,p)}$, as can be seen by contracting with the antiholomorphic $(0,4)$-form on X.

More explicitly, after gauge-fixing the kappa-symmetry, the physical fermion θ on the world-volume Σ can be expanded as

$$\theta = \epsilon \otimes P^+ \sum_{p=0}^{4} \Phi^{(p,0)}_{i_i \dots i_p} \gamma^{i_i \dots i_p} \xi \, , \tag{4.28}$$

where $\Phi^{(p,0)} \in \Lambda^{(p,0)}$ and ϵ is a two-component spinor in the noncompact directions. Expanding

$$\Phi^{(p,0)} = \widehat{\Phi}^{(p,0)} + \frac{1}{p} K^* \wedge \widehat{\Psi}^{(p-1,0)} \, , \tag{4.29}$$

where $\iota_K \widehat{\Phi}$, $\iota_K \widehat{\Psi} = 0$, and substituting P^+, (4.28) reads

$$\theta = \epsilon \otimes \Big(\widehat{\Phi}^{(0,0)} + \widehat{\Phi}^{(2,0)}_{ij} \gamma^{ij} + \widehat{\Phi}^{(1,0)}_{i} \gamma^{i} + \widehat{\Phi}^{(3,0)}_{ijk} \gamma^{ijk} \Big) \xi \, , \tag{4.30}$$

where we have set

$$\begin{aligned} \widehat{\Phi}^{(1,0)}_{i} &:= \widehat{\Psi}^{(0,0)} K^*_i \\ \widehat{\Phi}^{(3,0)}_{ijk} &:= \widehat{\Psi}^{(2,0)}_{[ij} K^*_{k]} \, . \end{aligned} \tag{4.31}$$

Equation (4.30) above is the explicit form of (4.27).

Zero modes. The zero modes on the fivebrane satisfy the Dirac equation (4.19) where, after gauge-fixing θ has positive chirality along the fivebrane world-volume, $\theta = P^+\theta$. Having explained the spinor-form correspondence, we would now like to rewrite the Dirac equation in terms of forms on the fivebrane. First, it would be useful to note the following relations:

$$\begin{aligned} (\Pi^{\|})^r_m F_{rnpq} \gamma^m \gamma^{npq} \theta_- &= 0 \\ (\Pi^{\|})^r_m F_{rnpq} \gamma^m \gamma^{npq} \theta_+ &= \frac{3}{4} F_{mnpq} \gamma^{mnpq} \theta_+ \, , \end{aligned} \tag{4.32}$$

where $\theta_\pm$ denotes the chirality of θ along the normal directions, and $\Pi^{\|}$ is the projector onto the fivebrane worldvolume defined in appendix B.1. Since θ has positive chirality along the fivebrane world-volume, we have $\theta_\pm = \frac{1}{2}(1 \pm \Gamma_9)\theta$. It further follows that

$$\mathcal{D}_m \theta^{(p,0)} = \epsilon \otimes \begin{cases} \nabla_m \widehat{\Phi}\xi \,, & p=0 \\ \nabla_m \widehat{\Phi}_r \gamma^r \xi - \frac{1}{4}\widehat{\Phi}^r F_{rstm}\gamma^{st}\xi \,, & p=1 \\ \nabla_m \widehat{\Phi}_{rs}\gamma^{rs}\xi - \frac{1}{6}\widehat{\Phi}^{rn} F_{rstm}\gamma^{st}{}_n\xi \,, & p=2 \\ \nabla_m \widehat{\Phi}_{rst}\gamma^{rst}\xi - \frac{3}{4}\widehat{\Phi}^{rnp} F_{rstm}\gamma^{st}{}_{np}\xi \,, & p=3 \end{cases} , \tag{4.33}$$

where we have denoted $\theta^{(p,0)} := \epsilon \otimes \widehat{\Phi}^{(p,0)}_{i_1 \dots i_p}\gamma^{i_1 \dots i_p}\xi$. Plugging (4.33) into (4.19), we obtain

$$\boxed{\begin{aligned} 0 &= \Big\{(\nabla^{\|})_m \widehat{\Phi} + 4(\nabla^{\|})^p \widehat{\Phi}_{pm}\Big\}\gamma^m \xi \\ 0 &= \Big\{(\nabla^{\|})_m \widehat{\Phi}_n + 6(\nabla^{\|})^p \widehat{\Phi}_{pmn} - \frac{1}{2}F_{mn}{}^{pq}\widehat{\Phi}_{pq}\Big\}\Omega^{mnrs}\gamma_{rs}\xi^* \\ 0 &= \Big\{(\nabla^{\|})_m \widehat{\Phi}_{np}\Big\}\Omega^{mnpq}\gamma_q \xi^* \\ 0 &= \Big\{(\nabla^{\|})_m \widehat{\Phi}_{npq}\Big\}\Omega^{mnpq} \,, \end{aligned}} \tag{4.34}$$

where $(\nabla^{\|})_m := (\Pi^{\|})_m^n \nabla_n$, is the covariant derivative projected along the fivebrane. Passing to complex coordinates, the above can be seen to be equivalent to equations (3.6-3.9) of [8], or (3.10-3.13) of [9].

Following the analysis of [9], the space of solutions to the above system of equations is spanned by harmonic forms[8] $\{\widehat{\Phi}^{(p,0)}_{I_p};\ p = 0 \dots 3\}$, where in addition the $\widehat{\Phi}^{(2,0)}$s satisfy the constraint

$$\mathcal{H}\Big\{F_{mnpq}\widehat{\Phi}^{np}(\Pi^{\|})^q_r dx^r\Big\} = 0 \tag{4.35}$$

and we have denoted by $\mathcal{H}$ the projector onto the space of harmonic forms. The corresponding fermion zero modes are of the form

$$\theta = \sum_{p=0}^{3}\sum_{I_p}\epsilon^{I_p} \otimes X_{I_p}\xi \,, \tag{4.36}$$

where (no summation over p)

$$X_{I_p} = \begin{cases} \widehat{\Phi}^{(p,0)}_{I_p}\gamma_{(p)}, & p \neq 2 \\ \widehat{\Phi}^{(2,0)}_{I_2}\gamma_{(2)} + \delta\widehat{\Phi}^{(1,0)}_{I_2}\gamma_{(1)} + \delta\widehat{\Phi}^{(3,0)}_{I_2}\gamma_{(3)}, & p = 2 \end{cases} ; \quad I_p = \begin{cases} 1, \dots, h^{p,0}(\Sigma), & p \neq 2 \\ 1, \dots, n, & p = 2 \end{cases} , \tag{4.37}$$

[8] The forms $\widehat{\Phi}^{(p,0)}_{I_p}$, $p = 1, 3$, have a leg in the normal bundle, see definition (4.31). More precisely: they are in $H^0(\Sigma, K \otimes \Omega^{3-p})$, $p = 1, 3$. Out of these, we can construct harmonic forms in $H^{0,p}(\Sigma) \cong H^p(\Sigma, \mathcal{O})$, by contracting with the antiholomorphic fourform on X. This is just the statement of Serre duality.

the $\widehat{\Phi}^{(p,0)}_{I_p}$s are harmonic and $\{\delta\widehat{\Phi}^{(1,0)}_{I_2},\ \delta\widehat{\Phi}^{(3,0)}_{I_2}\}$ is a special solution of the inhomogeneous equation

$$(\nabla^{\|})^{+}_{[m}\widehat{\Phi}_{n]} + 6(\nabla^{\|})^{p}\widehat{\Phi}_{pmn} = \frac{1}{2}F_{mn}{}^{pq}\widehat{\Phi}_{I_2,pq}\ . \tag{4.38}$$

In the above, n is the number of harmonic (2,0) forms on Σ which in addition satisfy the constraint (4.35); the ϵ^{I_p}s are spinors in the **2** of Spin(3) (after Wick-rotating to Euclidean signature). Note that (4.38) implies condition (4.35). The authors of [9] define a flux-dependent generalization of the arithmetic genus:

$$\chi_F := h^{0,0} - h^{1,0} + n - h^{3,0}\ . \tag{4.39}$$

5. Instanton contributions

We can now proceed to the computation of the instanton contributions to the coupling (1.8). The main result of the paper is arrived at in this section: instantons with four fermionic zeromodes do not contribute to the superpotential.

5.1 Gravitino Kaluza-Klein reduction

Before proceeding to integrate over the fermion zeromodes, we will need the Kaluza-Klein ansatz for the gravitino entering the vertex operator V in (3.5). As already discussed in the introduction, only terms which depend on the descendants of the linear multiplets contribute to the superpotential. Hence, the relevant part of the Kaluza-Klein ansatz for the gravitino reads

$$\begin{cases} \Psi_\mu = i(\omega_I\cdot J)\ \gamma_\mu\chi^I\otimes\xi^* + \text{c.c.} \\ \Psi_m = \chi^I\otimes\omega_{I,mp}\gamma^p\xi^* + \text{c.c.}\ ; \qquad I=1,\dots b_2\,, \end{cases} \tag{5.1}$$

where the χ^Is are complex spinors in the **2** of Spin(3), and $\omega_I\in H^2(X,\mathbb{R})$. As is straightforward to see, the eleven-dimensional gravitino equation, $\Gamma^M\mathcal{D}_{[M}\Psi_{N]}=0$, is satisfied if χ^I is a massless three-dimensional fermion,

$$\nabla\!\!\!\!/\,\chi^I = 0\,, \tag{5.2}$$

provided

$$\omega_I\lrcorner F = 0\ . \tag{5.3}$$

The implications of this condition were discussed extensively in the introduction. In this picture, χ^I is massless if it corresponds to a zero eigenvalue of the matrix T_{IJ} (in a diagonal basis). Alternatively this can be seen as follows. The quadratic part of the three-dimensional action for the χ^Is comes from the dimensional reduction of the quadratic-gravitino term in the eleven-dimensional supergravity action

$$\int d^{11}x\sqrt{g_{11}}\,\Psi_M\Gamma^{MNP}\mathcal{D}_N\Psi_P\ . \tag{5.4}$$

Plugging the Kaluza-Klein ansatz (5.1) in the action above, we obtain

$$\mathrm{Vol}(X)\int d^3x\sqrt{g_3}\Big(D_{IJ}\bar{\chi}^I\nabla\!\!\!/\,\chi^J-\frac{4}{9}T_{IJ}\bar{\chi}^I\chi^J\Big)\,, \tag{5.5}$$

where

$$D_{IJ}:=\int_X\Big(\omega_I\wedge\star\omega_J+\frac{2}{3}\,\omega_I\wedge\omega_J\wedge J\wedge J\Big) \tag{5.6}$$

and the Hodge star is with respect to the metric of the Calabi-Yau fourfold. In the above we have made use of the identity

$$\star(\omega_I\wedge\omega_J\wedge J\wedge J)=\frac{1}{2}\Big\{(\omega_I\cdot J)(\omega_J\cdot J)-2(\omega_I\cdot\omega_J)\Big\}\,, \tag{5.7}$$

which can be proven with the help of (B.8). As advertised, massless fermions correspond to zero eigenvalues of T_{IJ}.

We remark that in (5.5) there is no coupling of the form

$$\mathrm{Vol}(X)\int d^3x\sqrt{g_3}\Big(W_{IJ}\chi^I\chi^J+\mathrm{c.c.}\Big)\,. \tag{5.8}$$

In the following we will investigate whether such a term is generated by instanton contributions. In the context of three-dimensional supersymmetric field theory the fact that such a term can indeed be generated by instanton effects, was demonstrated in [49].

5.2 Two zeromodes

Before coming to the subject of instantons with four fermionic zeromodes in the next subsection, we will briefly comment on the case of instantons with two zeromodes (corresponding to the fivebrane wrapping rigid, isolated cycles). As can be seen from (4.36), there are always two zero modes corresponding to $p=0$:

$$\theta=\epsilon\otimes\xi\,. \tag{5.9}$$

These are the zero modes which come from the supersymmetry of the Calabi-Yau background.[9] We would like to compute the instanton contribution of these zeromodes to the superpotential. First, we need to define the integration over fermion zeromodes:

$$\int d^2\epsilon\;\epsilon^\alpha\epsilon^\beta:=C^{\alpha\beta}\,, \tag{5.10}$$

where C in the equation above is the charge-conjugation matrix in three dimensions. It follows that

$$\int d^2\epsilon\;(\chi\epsilon)(\epsilon\psi)=(\chi\psi)\,, \tag{5.11}$$

[9] In three-dimensional nomenclature the supersymmetry of the background is $\mathcal{N}=2$ (equivalently: $\mathcal{N}=1$ in four dimensions), i.e. four real supercharges. The instanton breaks half the supersymmetries, as can be seen from (4.22). Note that ξ in (5.9) is complex and ϵ is a spinor in the **2** of Spin(3). Henceforth we are complexifying our notation for θ, Ψ_m, V. At any rate, θ must be complexified in order to pass to Euclidean signature.

for any two three-dimensional spinors χ, ψ in the **2** of Spin(3). To simplify the presentation, we are using the notation $(\chi\psi) := (\chi^{Tr}C\psi)$.

Integrating over the zeromodes using the above prescription, we find that the instanton induces a two-fermion coupling of the form

$$\chi^I\chi^J \int [DZ'(\sigma)] v_I v_J e^{-S_{PST}[Z(\sigma);g,C,\Psi]} + \text{c.c.}\,, \tag{5.12}$$

where

$$v_I := 2i\int_\Sigma J\wedge J\wedge\omega_I \tag{5.13}$$

and the path integration above does not include the zeromodes. In (5.13) all the forms should be understood as pulled-back to Σ. In particular the pull-back of the almost complex structure to Σ can be identified with $\widehat{J}$, which is discussed from the point of view of the induced SU(3) structure on Σ in appendix B.1. Note that in the formula above the primitive part of ω_I is projected out.

We are not going to elaborate on the one-loop determinants, as this lies outside the main focus of this paper. The result of the integration over the bosonic coordinates should be obtainable using techniques similar to [12]. The integration over the fermionic variables is proportional to the determinant of the flux-dependent Dirac operator $\gamma^m\mathcal{D}^{\|}_m$ (away from its kernel), as follows from equation (3.8).

5.3 Four zeromodes

In the presence of four zeromodes there are the following possibilities which we will examine in turn: either $h^{0,0}=n=1$ (corresponding to $\chi_F=2$) or $h^{0,0}=h^{p,0}=1$, where p is odd (corresponding to $\chi_F=0$). Recall that n is the number of harmonic (2,0) forms on Σ which in addition satisfy the constraint (4.35). As we will see, no superpotential is generated in either case. Since $\chi_F\neq 1$ in all cases, we conclude that our result does not rule out the possibility that in the presence of flux the arithmetic genus criterion should be replaced by the condition $\chi_F=1$.

• $h^{0,0}=n=1$

In this case we have $\chi_F=2$. Let us substitute the Kaluza-Klein ansatz (5.1) and the expression for the zeromodes,

$$\theta=\epsilon\otimes\xi+\zeta\otimes\Big(\widehat{\Phi}_{mn}\gamma^{mn}+\delta\widehat{\Phi}_m\gamma^m+\delta\widehat{\Phi}_{mnp}\gamma^{mnp}\Big)\xi\,, \tag{5.14}$$

into equation (3.5) for the gravitino vertex operator. Integrating over the zeromodes using (5.10) we get, up to a total worldvolume derivative,

$$\int d^2\epsilon\; d^2\zeta\; VV=\chi^I\chi^J v_I w_J\,, \tag{5.15}$$

where v_I was defined in (5.13) above and

$$w_J := \frac{2}{9}\int_\Sigma \widehat{\Theta}\wedge\widehat{\Phi}\wedge\omega_J\;. \tag{5.16}$$

The object $\widehat{\Theta}$ is defined by

$$\widehat{\Theta}_{mn} := \Omega_{mnpq} F^{pq}{}_{rs} \widehat{\Phi}^{rs} \tag{5.17}$$

and is a (0,2)-form on Σ. (Recall that in our conventions Ω is antiholomorphic). In deriving this result, we had to perform some tedious but straightforward gamma-matrix algebra making repeated use of the formulæ in the appendices A), (B, especially equations (B.5), (B.7). Moreover we have taken into account the normal flux condition and we have implemented (1.10), as discussed in the introduction.

In the following we show that the right-hand side of (5.16) vanishes; no instanton-induced superpotential is generated in this case. Before demonstrating this fact however, let us note that the following group-theoretical reasoning can be used to gain insight into the result (5.15). As follows from the form of the vertex operator, the integration over the zeromodes receives three kinds of contributions:

$$\chi^I \chi^J v_I \otimes \omega_J \otimes F \otimes (\widehat{\Phi}^{(2,0)} + \delta\widehat{\Phi}^{(1,0)} + \delta\widehat{\Phi}^{(3,0)})^{2\otimes_s} , \tag{5.18}$$

coming from terms of the form $VV \propto (\Psi_m \Gamma^m \theta)(\Psi\theta^3)F$,

$$\chi^I \chi^J v_I \otimes \nabla\omega_J \otimes (\widehat{\Phi}^{(2,0)} + \delta\widehat{\Phi}^{(1,0)} + \delta\widehat{\Phi}^{(3,0)})^{2\otimes_s} , \tag{5.19}$$

coming from terms of the form $VV \propto (\Psi_m \Gamma^m \theta)(\nabla\Psi\theta^3)$, and

$$\chi^I \chi^J v_I \otimes \omega_J \otimes (\widehat{\Phi}^{(2,0)} + \delta\widehat{\Phi}^{(1,0)} + \delta\widehat{\Phi}^{(3,0)}) \otimes \nabla(\widehat{\Phi}^{(2,0)} + \delta\widehat{\Phi}^{(1,0)} + \delta\widehat{\Phi}^{(3,0)}) , \tag{5.20}$$

coming from terms of the form $VV \propto (\Psi_m \Gamma^m \theta)(\Psi\theta^2\nabla\theta)$. Contributions of the type (5.18) transform in the[10]

$$\Big((000) \oplus (101)\Big) \otimes (020) \otimes \Big((010) \oplus (100) \oplus (001)\Big)^{2\otimes_s}$$

of SU(4). There are exactly three scalars in the decomposition of the tensor product above. These we can write explicitly as:

$$\begin{aligned}
S_1 &:= \chi^I \chi^J v_I \omega_{J,mn} \Omega^{mpij} F_{ijqr} \widehat{\Phi}^{qr} \widehat{\Phi}_p{}^n \\
S_2 &:= \chi^I \chi^J v_I (\omega_J \cdot J) \Omega^{mpij} F_{ijqr} \widehat{\Phi}^{qr} \widehat{\Phi}_{mp} \\
S_3 &:= \chi^I \chi^J v_I \delta\widehat{\Phi}^i \delta\widehat{\Phi}^{jk}{}_m \Omega_{ijkn} F^{mnpq} \omega_{J,pq} \, .
\end{aligned} \tag{5.21}$$

The last one, however, vanishes by virtue of equation (5.3). Moreover, using equation (4.38), the scalars $S_{1,2}$ can be expressed as a linear combination of $R_1, \ldots R_7$ defined in equation (5.26) below:

$$\begin{aligned}
S_1 &= -2R_2 + 4R_5 - 4R_6 \\
S_2 &= 2R_4 - 8R_7 \, .
\end{aligned} \tag{5.22}$$

[10] In the following we are using the Dynkin notation for A_3.

In deriving the above we have used the identity

$$\delta\widehat{\Phi}_{qrs}\Omega^{rsmp} = -\frac{2}{3}\Omega^{ijk[m}(\Pi^+)_q{}^{p]}\delta\widehat{\Phi}_{ijk}\,, \tag{5.23}$$

which can be proved using (B.5). A direct computation of the terms of the form (5.18), yields the contribution

$$\frac{2i}{9}S_1 - \frac{1}{18}S_2 = -\frac{4i}{9}(R_2 - 2R_5 + 2R_6) - \frac{1}{9}(R_4 - 4R_7) \tag{5.24}$$

to the zeromode integral (5.15). The linear combination above can be written in a more elegant way by noting that

$$iS_1 - \frac{1}{4}S_2 = \chi^I\chi^J v_I \star(\widehat{\Theta}\wedge\widehat{\Phi}\wedge\omega_J)\,, \tag{5.25}$$

where the Hodge star is along Σ. In proving (5.25) we have made use of equation (B.18).

Taking into account that ω_I is a harmonic (1,1) form and that therefore $(\omega_I\cdot J)$ is a constant,[11] it follows that $\nabla\omega_I$ transforms in the $(201)\oplus(102)$ of SU(4). Hence, contributions of the type (5.19) transform in the

$$\Big((201)\oplus(102)\Big)\otimes\Big((010)\oplus(100)\oplus(001)\Big)^{2\otimes_s}$$

of SU(4). As there are no scalars in the decomposition of the tensor product above, we conclude that these terms vanish.

Taking into account that $\widehat{\Phi}^{(2,0)}$ is a harmonic (2,0) form on a Kähler manifold, it follows that $\nabla\widehat{\Phi}^{(2,0)}$ transforms in the (110) of SU(4). Similarly, $\nabla\delta\widehat{\Phi}^{(1,0)}$ transforms in the $(000)\oplus(200)\oplus(010)\oplus(101)$ of SU(4). Finally, taking into account the last of equations (4.34), it follows that $\nabla\delta\widehat{\Phi}^{(3,0)}$ transforms in the $(010)\oplus(101)\oplus(002)$ of SU(4). Putting everything together, it follows that contributions of the type (5.20) transform in the

$$\Big((000)\oplus(101)\Big)\otimes\Big((010)\oplus(100)\oplus(001)\Big)$$
$$\otimes\Big((110)\oplus(000)\oplus(200)\oplus 2(010)\oplus 2(101)\oplus(002)\Big)$$

of SU(4). There are exactly seven scalars in the decomposition of the tensor product above: one coming from $\nabla\widehat{\Phi}^{(2,0)}$, three from $\nabla\delta\widehat{\Phi}^{(1,0)}$ and three from $\nabla\delta\widehat{\Phi}^{(3,0)}$. These can

[11] A direct computation reveals that it is in fact $\widehat{\omega}^I$ rather than ω_I, where the hat denotes the pull-back to Σ, which appears in the various invariants of this section. However, using the inclusion map

$$\iota^*:\; H^{p,q}(X,\mathbb{R})\longrightarrow H^{p,q}(\Sigma,\mathbb{R})\,,$$

we can think of ω^I as the extension to X of the harmonic form $\widehat{\omega}^I$ on Σ [50]. In the text, we do not make an explicit distinction between ω_I and $\widehat{\omega}^I$. See also the next footnote.

be written explicitly as

$$
\begin{aligned}
R_1 &:= \chi^I \chi^J v_I \nabla^m \widehat{\Phi}_{ij} \Omega^{ijpq} \delta\widehat{\Phi}_p \omega_{J,qm} \\
R_2 &:= \chi^I \chi^J v_I \nabla_m \delta\widehat{\Phi}_n \Omega^{mnij} \omega_{J,ip} \widehat{\Phi}^p{}_j \\
R_3 &:= \chi^I \chi^J v_I \nabla^m \delta\widehat{\Phi}_n \Omega^{nijk} \omega_{J,km} \widehat{\Phi}_{ij} \\
R_4 &:= \chi^I \chi^J v_I (\omega_J \cdot J) \nabla_m \delta\widehat{\Phi}_n \Omega^{mnij} \widehat{\Phi}_{ij} \\
R_5 &:= \chi^I \chi^J v_I \nabla^m \delta\widehat{\Phi}_{ijk} \Omega^{ijkq} \omega_{J,qp} \widehat{\Phi}^p{}_m \\
R_6 &:= \chi^I \chi^J v_I \nabla^m \delta\widehat{\Phi}_{ijk} \Omega^{ijkq} \omega_{J,mp} \widehat{\Phi}^p{}_q \\
R_7 &:= \chi^I \chi^J v_I (\omega_J \cdot J) \nabla^m \delta\widehat{\Phi}_{ijk} \Omega^{ijkq} \widehat{\Phi}_{qm} \ .
\end{aligned}
\tag{5.26}
$$

A direct computation of the terms of the form (5.20), yields the contribution

$$
-4i(R_1 + R_3 + 2R_5) \tag{5.27}
$$

to the zeromode integral (5.15).

Putting the contributions (5.24), (5.27) together, we arrive at equation (5.15). Note that the invariants $R_4, \dots R_7$ as well as the linear combinations $R_1 + 2R_2$ and $R_1 + R_3$, can be written as total derivatives. This can readily be seen by taking into account that Ω is covariantly constant while ω, $\widehat{\Phi}$ are harmonic.[12] It follows that the total contribution can be cast in the form $\propto R_2$+total derivative. On the other hand, up to a total derivative, R_2 is proportional to the right-hand-side of (5.25), as follows from (5.24), (5.25).

We are now ready to show that the left-hand-side of (5.16) vanishes identically. First note that, as follows from (4.35) or (4.38), the projection of $\widehat{\Theta}$ onto the space of harmonic forms on Σ vanishes: $\mathcal{H}\{\widehat{\Theta}\} = 0$. It follows that

$$
\int_\Sigma \widehat{\Theta} \wedge \widehat{\Phi} \wedge J = 0 \,, \tag{5.28}
$$

since $\widehat{\Phi} \wedge J$ is harmonic (this can be seen by noting that $\star\widehat{\Phi} = \widehat{\Phi} \wedge J$). Varying this equation with respect to the Kähler structure, $\phi^I \to \phi^I + \delta\phi^I$, we get

$$
\int_\Sigma \frac{\delta\widehat{\Theta}}{\delta\phi^I} \wedge \widehat{\Phi} \wedge J + \int_\Sigma \widehat{\Theta} \wedge \widehat{\Phi} \wedge \omega_I = 0 \ . \tag{5.29}
$$

Furthermore, under a Kähler-structure variation the metric transforms as

$$
\delta g_{mn} = \sum_I \delta\phi^I \omega_{I,mp} J_n{}^p \ . \tag{5.30}
$$

---

[12] Note that in general the pull-back of the Christoffel connection from the total space X to the base Σ, $(\nabla^{\|})_m$, *cannot* be identified with the Christoffel connection $\widehat{\nabla}_m$ associated with the metric on Σ. However if $\widehat{S}$ is an arbitrary p-form on Σ whose extension to X is S, we have

$$
(\nabla^{\|})_m S^{mm_2 \dots m_p} = \nabla_m S^{mm_2 \dots m_p} = (\widehat{\nabla})_m \widehat{S}^{mm_2 \dots m_p} \ .
$$

The first equality follows from (B.21). The second equality follows from $\Gamma^n_{mn} = g^{-1/2}\partial_m g^{1/2}$ and the fact that the determinants of the metrics X, Σ are equal, as can be seen from the explicit form of the fibration (B.19).

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Note that the right-hand side above is automatically symmetric in the indices m, n. Taking the above into account together with the fact that S_2 is a total worldvolume derivative it follows that

$$\int_\Sigma \frac{\delta\widehat{\Theta}}{\delta\phi^I} \wedge \widehat{\Phi} \wedge J = 0 \ . \tag{5.31}$$

In the derivation we made use of the identity

$$\widehat{\Phi}^{mn}\Omega_{mnpq}\widehat{\Phi}^{rs}F_{rs}{}^{qt}\omega_{I,pt} = -\widehat{\Phi}^{mn}\Omega_{mn}{}^{pq}\widehat{\Phi}_s{}^t F_{pq}{}^{sr}\omega_{I,rt} \ . \tag{5.32}$$

From (5.29), (5.31) it finally follows that the right-hand side of (5.16) vanishes, as advertised.

No potential is generated in the remaining cases either, as we now show.

- $h^{0,0} = h^{1,0} = 1$.

  In this case we have $\chi_F = 0$. As can be verified by direct computation, no potential is generated in this case. The easiest way to arrive at this result is by the following group-theoretical argument. It follows from the form of the vertex operator that the integration over the zeromodes receives three kinds of contributions:

  $$\chi^I\chi^J v_I \otimes \omega_J \otimes F \otimes \widehat{\Phi}^{(1,0)} \otimes \widehat{\Phi}^{(1,0)} \ , \tag{5.33}$$

  coming from terms of the form $VV \propto (\Psi_m\Gamma^m\theta)(\Psi\theta^3)F$,

  $$\chi^I\chi^J v_I \otimes \nabla\omega_J \otimes \widehat{\Phi}^{(1,0)} \otimes \widehat{\Phi}^{(1,0)} \ , \tag{5.34}$$

  coming from terms of the form $VV \propto (\Psi_m\Gamma^m\theta)(\nabla\Psi\theta^3)$, and

  $$\chi^I\chi^J v_I \otimes \omega_J \otimes \widehat{\Phi}^{(1,0)} \otimes \nabla\widehat{\Phi}^{(1,0)} \ , \tag{5.35}$$

  coming from terms of the form $VV \propto (\Psi_m\Gamma^m\theta)(\Psi\theta^2\nabla\theta)$. Contributions of the type (5.33) transform in the

  $$\Big((000) \oplus (101)\Big) \otimes (020) \otimes (100)^{2\otimes_s}$$

  of SU(4). As there are no scalars in the decomposition of the tensor product above, we conclude that these terms vanish.

  Taking into account that ω_I is a harmonic (1,1) form, it follows that $\nabla\omega_I$ transforms in the $(201) \oplus (102)$ of SU(4). Hence, contributions of the type (5.34) transform in the

  $$\Big((201) \oplus (102)\Big) \otimes (100)^{2\otimes_s}$$

  of SU(4). As there are no scalars in the decomposition of the tensor product above, we conclude that these terms vanish.

  Taking into account that $\widehat{\Phi}^{(1,0)}$ is harmonic, it follows that $\widehat{\Phi}^{(1,0)}$ transforms in the (200) of SU(4). Hence, contributions of the type (5.35) transform in the

  $$\Big((000) \oplus (101)\Big) \otimes (100) \otimes (200)$$

  of SU(4). As there are no scalars in the decomposition of the tensor product above, we conclude that these terms vanish.

- $h^{0,0} = h^{3,0} = 1$.

  In this case we have $\chi_F = 0$. As in the previous case, no potential is generated. This can be shown e.g. by the same type of group-theoretical reasoning as before.

6. Discussion

Taking advantage of the recent progress in explicit theta-expansions in eleven-dimensional superspace [17], we have performed a computation of the contribution of fivebrane instantons with four fermionic zeromodes in M-theory compactifications on Calabi-Yau fourfolds with (normal) flux. The calculus of fivebrane instantons in M-theory is still largely unexplored, and we hope that our computation will initiate a more extensive study of these phenomena directly in M-theory.

We have found that no superpotential is generated in this case — a result which is compatible with replacing the arithmetic genus criterion by the condition $\chi_F = 1$, where χ_F is the flux-dependent 'index' of [9]. It would be interesting to reexamine this statement when the condition of normal flux is relaxed.

It would be desirable to explore the obvious generalizations of our computation: fivebrane instanton contributions to non-holomorphic couplings, and/or contributions to higher-derivative and multi-fermion couplings as in [10]. The expansions of [17] can also be used to study instantons with more than four zeromodes.

So far the precise relation between instanton calculus in M-theory [11, 12] and the rules of D-instanton computations in string theory put forward in [51 − 53], has not been clearly spelled out. Understanding this relation may help clarify some of the conceptual issues associated with the M-theory calculus, see e.g. [12]. This would be another interesting possibility for future investigation.

Last but not least, it is important to address the reservations, discussed in the introduction, about the fivebrane action of [33] and to incorporate the topological considerations of [37, 38] in a supersymmetric context.[13]

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A. Gamma-matrix identities

The gamma matrices in eight dimensions have the following properties

Symmetry:

$$(C\gamma_{(n)})^{Tr} = (-)^{\frac{1}{2}n(n-1)} C\gamma_{(n)} \,, \tag{A.1}$$

[13] I would like to thank Greg Moore for correspondence on this point.

where C is the charge-conjugation matrix.

Hodge duality:

$$\star\gamma_{(n)} = (-)^{\frac{1}{2}n(n+1)}\gamma_{(8-n)}\gamma_9\,, \tag{A.2}$$

where γ_9 is the chirality matrix.

Complex conjugation:

$$\gamma^*_{(n)} = C\gamma_{(n)}C^{-1}\ . \tag{A.3}$$

Reality:

A Majorana-Weyl spinor $\xi_\pm$ in eight dimensions, where the subscript denotes the chirality, satisfies

$$\xi^\dagger_\pm = \pm\xi^{Tr}_\pm C\,, \tag{A.4}$$

which, together with the complex conjugation above, imply

$$(\gamma_{(n)}\xi_\pm)^* = \pm C\gamma_{(n)}\xi_\pm\ . \tag{A.5}$$

Decomposition $10 \to 3+8$:

We decompose the eleven-dimensional matrices as follows:

$$\begin{aligned}\Gamma^\mu &= \gamma^\mu \otimes \gamma_9\ , \qquad \mu = 0,1,2\\ \Gamma^m &= 1_2 \otimes \gamma^m\,, \qquad m = 3,\ldots 10\ .\end{aligned} \tag{A.6}$$

The eleven-dimensional charge-conjugation matrix decomposes as

$$C_{11} = C_3 \otimes C_8\gamma_9\,, \tag{A.7}$$

where γ_9 is the chirality matrix in eight dimensions and C_3, C_8 are the charge-conjugation matrices in three, eight dimensions respectively.

B. SU(4) structure

The existence of a nowhere-vanishing positive-chirality complex spinor ξ on X, implies the reduction of the structure group to SU(4). The SU(4) structure can be equivalently specified in terms of a complex self-dual fourform Ω and an almost complex structure J satisfying

$$\begin{aligned}J\wedge\Omega &= 0\\ \Omega\wedge\Omega^* &= \frac{2}{3}J^4\ .\end{aligned} \tag{B.1}$$

In terms of ξ bilinears we have

$$\begin{aligned}J_{mn} &= i\xi^\dagger\gamma_{mn}\xi\\ \Omega_{mnpq} &= \xi^{Tr}\gamma_{mnpq}\xi\end{aligned} \tag{B.2}$$

and we have normalized $\xi^\dagger\xi = 1$. Using the almost complex structure we can define the projectors

$$(\Pi^{\pm})_m{}^n := \frac{1}{2}(\delta_m{}^n \mp iJ_m{}^n) \tag{B.3}$$

with respect to which Ω is antiholomorphic

$$(\Pi^-)_m{}^i\Omega_{inpq} = \Omega_{mnpq}\ ; \qquad (\Pi^+)_m{}^i\Omega_{inpq} = 0\ . \tag{B.4}$$

The following useful identities can be proved e.g. by Fierzing

$$\begin{aligned}
\frac{1}{4!\times 2^4}\ \Omega_{rstu}\Omega^{*rstu} &= 1\\
\frac{1}{6\times 2^4}\ \Omega_{irst}\Omega^{*mrst} &= (\Pi^-)_i{}^m\\
\frac{1}{4\times 2^4}\ \Omega_{ijrs}\Omega^{*mnrs} &= (\Pi^-)_{[i}{}^m(\Pi^-)_{j]}{}^n\\
\frac{1}{6\times 2^4}\ \Omega_{ijkr}\Omega^{*mnpr} &= (\Pi^-)_{[i}{}^m(\Pi^-)_j{}^n(\Pi^-)_{k]}{}^p\\
\frac{1}{4!\times 2^4}\ \Omega_{ijkl}\Omega^{*mnpq} &= (\Pi^-)_{[i}{}^m(\Pi^-)_j{}^n(\Pi^-)_k{}^p(\Pi^-)_{l]}{}^q\,,
\end{aligned} \tag{B.5}$$

and

$$\begin{aligned}
\gamma_m\xi &= (\Pi^-)_m{}^n\gamma_n\xi\\
\gamma_{mn}\xi &= -iJ_{mn}\xi - \frac{1}{8}\Omega_{mnpq}\gamma^{pq}\xi^*\\
\gamma_{mnp}\xi &= -3iJ_{[mn}\gamma_{p]}\xi - \frac{1}{2}\Omega_{mnpq}\gamma^q\xi^*\\
\gamma_{mnpq}\xi &= -3J_{[mn}J_{pq]}\xi + \frac{3i}{4}J_{[mn}\Omega_{pq]ij}\gamma^{ij}\xi^* + \Omega_{mnpq}\xi^*\ .
\end{aligned} \tag{B.6}$$

The action of $\gamma_{m_1\ldots m_p}$, $p \geq 5$, on ξ can be related to the above formulæ, using the Hodge properties of gamma matrices given in appendix A. From the above it follows that

$$\begin{aligned}
\xi^\dagger\xi &= 1; &\quad \xi^{Tr}\xi &= 0\\
\xi^\dagger\gamma_{mn}\xi &= -iJ_{mn}; &\quad \xi^{Tr}\gamma_{mn}\xi &= 0\\
\xi^\dagger\gamma_{mnpq}\xi &= -3J_{[mn}J_{pq]}; &\quad \xi^{Tr}\gamma_{mnpq}\xi &= \Omega_{mnpq}\\
\xi^\dagger\gamma_{mnpqrs}\xi &= 15iJ_{[mn}J_{pq}J_{rs]}; &\quad \xi^{Tr}\gamma_{mnpqrs}\xi &= 0\\
\xi^\dagger\gamma_{mnpqrstu}\xi &= 105J_{[mn}J_{pq}J_{rs}J_{tu]}; &\quad \xi^{Tr}\gamma_{mnpqrstu}\xi &= 0\,,
\end{aligned} \tag{B.7}$$

where we have made use of the identities

$$\begin{aligned}
\varepsilon_{mnpqrstu}J^{rs}J^{tu} &= 24J_{[mn}J_{pq]}\\
\varepsilon_{mnpqrstu}J^{tu} &= 30J_{[mn}J_{pq}J_{rs]}\\
\varepsilon_{mnpqrstu} &= 105J_{[mn}J_{pq}J_{rs}J_{tu]}\ .
\end{aligned} \tag{B.8}$$

Note that the bilinears $\xi^{Tr}\gamma_{(p)}\xi$, $\xi^\dagger\gamma_{(p)}\xi$, vanish for p odd. Finally, the last line of equation (B.5) together with the last line of the equation above imply

$$\Omega_{[ijkl}\Omega^*_{mnpq]} = \frac{8}{35}\varepsilon_{ijklmnpq}\ . \tag{B.9}$$

B.1 SU(4) vs SU(3)

In the case where there exists a nowhere-vanishing complex vector K, one can construct a corresponding nowhere-vanishing negative-chirality complex spinor

$$\xi_- := K^m \gamma_m \xi \ . \tag{B.10}$$

This implies the reduction of the structure group of X to SU(3). Without loss of generality we can take K to be antiholomorphic with respect to the almost complex structure J,

$$(\Pi^-)_m{}^n K_n = K_m; \qquad (\Pi^+)_m{}^n K_n = 0 \,, \tag{B.11}$$

and to satisfy

$$K_m K^m = 0; \qquad K^*_m K^m = 2 \ . \tag{B.12}$$

The SU(3) structure can be given in terms of an antiholomorphic (0,3) form $\widehat{\Omega}$ and a (1,1) form $\widehat{J}$ defined by

$$\begin{aligned} J &= \widehat{J} - \frac{i}{2} K \wedge K^* \\ \Omega &= iK \wedge \widehat{\Omega} \,, \end{aligned} \tag{B.13}$$

which satisfy $\iota_K \widehat{J}, \iota_K \widehat{\Omega}, \iota_{K^*} \widehat{\Omega} = 0$. Moreover, we can define antiholomorphic projectors $(\widehat{\Pi})$ with respect to the structure $\widehat{J}$:

$$(\Pi^-)_m{}^n = (\widehat{\Pi}^-)_m{}^n + \frac{1}{2} K_m K^{*n} \ . \tag{B.14}$$

The complex vector K specifies an almost product structure given by

$$R_m{}^n := K_m K^{*n} + K^*_m K^n - \delta_m{}^n \ . \tag{B.15}$$

Moreover every tensor can be decomposed into directions along and perpendicular to the K-orthogonal subspaces, using the projectors

$$\begin{aligned} (\Pi^{\|})_m^n &:= \delta_m^n - \frac{1}{2}(K_m K^{*n} + K^*_m K^n) \\ (\Pi^{\perp})_m^n &:= \frac{1}{2}(K_m K^{*n} + K^*_m K^n) \ . \end{aligned} \tag{B.16}$$

In particular, the metric decomposes as

$$g_{mn} = G_{mn} + \frac{1}{2}(K_m K^*_n + K^*_m K_n) \,, \tag{B.17}$$

where $K^m G_{mn} = 0$. Finally, note the useful identity

$$\varepsilon^{\|}_{mnpqrs} = -15 \widehat{J}_{[mn} \widehat{J}_{pq} \widehat{J}_{rs]} \,, \tag{B.18}$$

which can be proven by contracting the last line of (B.8) with $K^t K^{*u}$.

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(Anti)holomorphic six-cycles. We shall be interested in particular in the case where the almost product structure on X, defined in the previous section, is integrable and the metric on X can be brought to the standard form of a fibration over a six-cycle Σ:

$$ds^2(X) = G_{mn}(x)dx^m \otimes dx^n + (dz + A) \otimes (dz^* + A^*) \ . \tag{B.19}$$

This is of the form (B.17), where $K^* = dz + A(x)$ is the one-form dual of the holomorphic Killing vector $\partial/\partial z$, the x^ms are the coordinates on the base, z is a complex coordinate on the normal fibre and $A(x)$ is a complex connection one-form. The six-cycle defined by the fibration is a holomorphic cycle, with a similar definition for an antiholomorphic cycle. We would like to stress that in general X is not the total space of the normal bundle over Σ, but this approximation becomes more accurate as the size of Σ is scaled up.

Note that for any S_m such that $S_m = (\Pi^\perp)_m^q S_q$, we have

$$\nabla^{\parallel}_m S^m = (\Pi^{\parallel})_m^q \nabla_q (\Pi^\perp)_n^m S^n = \frac{1}{2}(\Pi^{\parallel})^{mq}(K_n \nabla_q K^*_m + \text{c.c.})S^n = 0 \ . \tag{B.20}$$

In the first equality we have used the orthogonality of $\Pi^\perp$, $\Pi^{\parallel}$. In the second equality we have taken (B.16) into account, and we have noted that $K^m(\Pi^{\parallel})_m^n = 0$. In the last equality we have taken into account that $(\Pi^{\parallel})^{mn}$ is symmetric, and that K is Killing. Similarly we can prove that if $S_m = (\Pi^{\parallel})_m^q S_q$, we have

$$\nabla^{\perp}_m S^m = 0 \ . \tag{B.21}$$

Equations (B.20), (B.21) can also be generalized to p-forms.

C. Gravitino vertex operator

In this appendix we give the details of the derivation of the gravitino vertex operator of section 3.1, equation (3.5).

For any Q, let ΔQ the part of Q linear in the gravitino. From (3.2) and the analysis of the previous sections we find

$$\begin{aligned} g_{mn} &= g^{(0)}_{mn} + g^{(1)}_{mn} + g^{(2)}_{mn} + \mathcal{O}(\Psi, \theta^3) \\ \Delta g_{mn} &= \Delta g^{(1)}_{mn} + \Delta g^{(2)}_{mn} + \Delta g^{(3)}_{mn} + \mathcal{O}(\Psi^2, \theta^5) \,, \end{aligned} \tag{C.1}$$

where

$$\begin{aligned} g^{(0)}_{mn} &= G_{mn} \\ g^{(1)}_{mn} &= -i(\mathcal{D}_{(m}\theta\Gamma_{n)}\theta) \\ g^{(2)}_{mn} &= -\frac{1}{4}(\mathcal{D}_m\theta\Gamma^{\underline{a}}\theta)(\mathcal{D}_n\theta\Gamma_{\underline{a}}\theta) \\ \Delta g^{(1)}_{mn} &= -2i(\Psi_{(m}\Gamma_{n)}\theta) \\ \Delta g^{(2)}_{mn} &= -(\Psi_{(m}\Gamma^{\underline{a}}\theta)(\mathcal{D}_{n)}\theta\Gamma_{\underline{a}}\theta) \\ \Delta g^{(3)}_{mn} &= \frac{1}{3}(\Psi_{(m}\mathfrak{G}\Gamma_{n)}\theta) + \frac{1}{3}(\Psi_{\underline{pq}}\mathcal{I}_{(m}{}^{\underline{pq}}\Gamma_{n)}\theta) \end{aligned} \tag{C.2}$$

For the inverse of the Green-Schwarz metric we have

$$
\begin{aligned}
g^{mn} &= g^{mn}_{(0)} + g^{mn}_{(1)} + g^{mn}_{(2)} + \mathcal{O}(\Psi,\theta^3)\\
\Delta g^{mn} &= \Delta g^{mn}_{(1)} + \Delta g^{mn}_{(2)} + \Delta g^{mn}_{(3)} + \mathcal{O}(\Psi^2,\theta^5)\,,
\end{aligned} \tag{C.3}
$$

where

$$
\begin{aligned}
g^{mn}_{(0)} &= G^{mn}\\
g^{mn}_{(1)} &= i(\mathcal{D}^{(m}\theta\Gamma^{n)}\theta)\\
g^{mn}_{(2)} &= \frac{1}{4}(\mathcal{D}^m\theta\Gamma^{\underline{a}}\theta)(\mathcal{D}^n\theta\Gamma_{\underline{a}}\theta) - \frac{1}{4}(\mathcal{D}^m\theta\Gamma^p\theta)(\mathcal{D}^n\theta\Gamma_p\theta)\\
&\quad - \frac{1}{4}(\mathcal{D}_p\theta\Gamma^m\theta)(\mathcal{D}^p\theta\Gamma^n\theta) - \frac{1}{2}(\mathcal{D}_p\theta\Gamma^{(m}\theta)(\mathcal{D}^{n)}\theta\Gamma^p\theta)\\
\Delta g^{mn}_{(1)} &= -\,G^{mk}\Delta g^{(1)}_{kl}G^{ln}\\
\Delta g^{mn}_{(2)} &= -\,G^{mk}\Delta g^{(2)}_{kl}G^{ln} - 2g^{k(m}_{(1)}\Delta g^{(1)}_{kl}G^{n)l}\\
\Delta g^{mn}_{(3)} &= -\,G^{mk}\Delta g^{(3)}_{kl}G^{ln} - 2g^{k(m}_{(1)}\Delta g^{(2)}_{kl}G^{n)l}\\
&\quad - 2g^{k(m}_{(2)}\Delta g^{(1)}_{kl}G^{n)l} - g^{mk}_{(1)}\Delta g^{(1)}_{kl}g^{ln}_{(1)}\ .
\end{aligned} \tag{C.4}
$$

For later use note that

$$
\frac{1}{\sqrt{-g}} = \Big(\frac{1}{\sqrt{-g}}\Big)^{(0)} + \Big(\frac{1}{\sqrt{-g}}\Big)^{(1)} + \Big(\frac{1}{\sqrt{-g}}\Big)^{(2)} + \mathcal{O}(\Psi,\theta^3)\,, \tag{C.5}
$$

where

$$
\begin{aligned}
\Big(\frac{1}{\sqrt{-g}}\Big)^{(0)} &= \frac{1}{\sqrt{-G}}\\
\Big(\frac{1}{\sqrt{-g}}\Big)^{(1)} &= \frac{i}{2\sqrt{-G}}(\mathcal{D}_m\theta\Gamma^m\theta)\\
\Big(\frac{1}{\sqrt{-g}}\Big)^{(2)} &= -\frac{1}{8\sqrt{-G}}\Big\{(\mathcal{D}_m\theta\Gamma^m\theta)^2 - (\mathcal{D}_m\theta\Gamma^{\underline{a}}\theta)(\mathcal{D}^m\theta\Gamma_{\underline{a}}\theta)\\
&\qquad + (\mathcal{D}_m\theta\Gamma^p\theta)(\mathcal{D}^m\theta\Gamma_p\theta) + (\mathcal{D}_m\theta\Gamma^p\theta)(\mathcal{D}_p\theta\Gamma^m\theta)\Big\}\ .
\end{aligned} \tag{C.6}
$$

Moreover

$$
\begin{aligned}
H_{mnp} &= H^{(0)}_{mnp} + H^{(1)}_{mnp} + H^{(2)}_{mnp} + \mathcal{O}(\Psi,\theta^3)\\
\Delta H_{mnp} &= \Delta H^{(1)}_{mnp} + \Delta H^{(2)}_{mnp} + \Delta H^{(3)}_{mnp} + \mathcal{O}(\Psi^2,\theta^5)\,,
\end{aligned} \tag{C.7}
$$

where

$$
\begin{aligned}
H^{(0)}_{mnp} &= F_{mnp} - c_{mnp} \\
H^{(1)}_{mnp} &= \frac{3i}{2}(\mathcal{D}_{[m}\theta\Gamma_{np]}\theta) \\
H^{(2)}_{mnp} &= \frac{3}{4}(\mathcal{D}_{[m}\theta\Gamma_{n}{}^{\underline{a}}\theta)(\mathcal{D}_{p]}\theta\Gamma_{\underline{a}}\theta) \\
\Delta H^{(1)}_{mnp} &= 3i(\Psi_{[m}\Gamma_{np]}\theta) \\
\Delta H^{(2)}_{mnp} &= (\Psi_{[m}\Gamma_{n}{}^{\underline{a}}\theta)(\mathcal{D}_{p]}\theta\Gamma_{\underline{a}}\theta) + 2(\Psi_{[m}\Gamma^{\underline{a}}\theta)(\mathcal{D}_{n}\theta\Gamma_{p]\underline{a}}\theta) \\
\Delta H^{(3)}_{mnp} &= -\frac{1}{2}(\Psi_{[m}\mathfrak{G}\Gamma_{np]}\theta) - \frac{1}{2}(\Psi_{\underline{pq}}\mathcal{I}_{[m}{}^{\underline{pq}}\Gamma_{np]}\theta)\ .
\end{aligned}
\tag{C.8}
$$

Similarly

$$
\begin{aligned}
v_p &= v^{(0)}_p + v^{(1)}_p + v^{(2)}_p + \mathcal{O}(\Psi,\theta^3) \\
\Delta v_p &= \Delta v^{(1)}_p + \Delta v^{(2)}_p + \Delta v^{(3)}_p + \mathcal{O}(\Psi^2,\theta^5)\,,
\end{aligned}
\tag{C.9}
$$

where

$$
\begin{aligned}
v^{(0)}_p &= a_p \\
v^{(1)}_p &= \frac{i}{2}(\mathcal{D}^m\theta\Gamma^n\theta)a_m a_n a_p \\
v^{(2)}_p &= \frac{1}{8}a_p a_m a_n\Big\{(\mathcal{D}^m\theta\Gamma_{\underline{a}}\theta)(\mathcal{D}^n\theta\Gamma^{\underline{a}}\theta) - (\mathcal{D}^m\theta\Gamma_p\theta)(\mathcal{D}^n\theta\Gamma^p\theta) - (\mathcal{D}_p\theta\Gamma^m\theta)(\mathcal{D}^p\theta\Gamma^n\theta) \\
&\qquad - 2(\mathcal{D}_p\theta\Gamma^m\theta)(\mathcal{D}^n\theta\Gamma^p\theta) - 3(\mathcal{D}^m\theta\Gamma^n\theta)(\mathcal{D}^q\theta\Gamma^r\theta)a_q a_r\Big\} \\
\Delta v^{(1)}_p &= \frac{1}{2}a_p a_k a_l \Delta g^{kl}_{(1)} \\
\Delta v^{(2)}_p &= \frac{1}{2}v^{(1)}_p a_k a_l \Delta g^{kl}_{(1)} + a_p a_k v^{(1)}_l \Delta g^{kl}_{(1)} + \frac{1}{2}a_p a_k a_l \Delta g^{kl}_{(2)} \\
\Delta v^{(3)}_p &= \frac{1}{2}v^{(2)}_p a_k a_l \Delta g^{kl}_{(1)} + a_p a_k v^{(2)}_l \Delta g^{kl}_{(1)} + v^{(1)}_p v^{(1)}_k a_l \Delta g^{kl}_{(1)} + \frac{1}{2}a_p v^{(1)}_k v^{(1)}_l \Delta g^{kl}_{(1)} \\
&\quad + \frac{1}{2}v^{(1)}_p a_k a_l \Delta g^{kl}_{(2)} + a_p a_k v^{(1)}_l \Delta g^{kl}_{(2)} + \frac{1}{2}a_p a_k a_l \Delta g^{kl}_{(3)} \\
a_p &:= \frac{\partial_p a}{\sqrt{-G^{mn}\partial_m a\partial_n a}}\ .
\end{aligned}
\tag{C.10}
$$

Also

$$
\begin{aligned}
\widetilde{H}_{mn} &= \widetilde{H}^{(0)}_{mn} + \widetilde{H}^{(1)}_{mn} + \widetilde{H}^{(2)}_{mn} + \mathcal{O}(\Psi,\theta^3) \\
\Delta\widetilde{H}_{mn} &= \Delta\widetilde{H}^{(1)}_{mn} + \Delta\widetilde{H}^{(2)}_{mn} + \Delta\widetilde{H}^{(3)}_{mn} + \mathcal{O}(\Psi^2,\theta^5)\,,
\end{aligned}
\tag{C.11}
$$

where

$$\begin{aligned}
\widetilde{H}^{(k)}_{mn} &= \frac{1}{6}\epsilon^{m'n'pqrs}\Big(\frac{1}{\sqrt{-g}}g_{mm'}g_{nn'}v_pH_{qrs}\Big)^{(k)} \\
\Delta\widetilde{H}^{(k)}_{mn} &= \frac{1}{6}\epsilon^{m'n'pqrs}\sum_{i=1}^{k}\Big\{\Big(\frac{1}{\sqrt{-g}}g_{mm'}g_{nn'}H_{qrs}\Big)^{(k-i)}\Delta v^{(i)}_p \\
&\quad + \Big(\frac{1}{\sqrt{-g}}g_{mm'}g_{nn'}v_p\Big)^{(k-i)}\Delta H^{(i)}_{qrs} - \frac{1}{2}\Big(\frac{1}{\sqrt{-g}}g_{mm'}g_{nn'}g^{kl}v_pH_{qrs}\Big)^{(k-i)}\Delta g^{(i)}_{kl} \\
&\quad + 2\Delta g^{(i)}_{[m|m'}\Big(\frac{1}{\sqrt{-g}}g_{|n]n'}v_pH_{qrs}\Big)^{(k-i)}\Big\}, \quad k=1,2,3\ . \qquad \text{(C.12)}
\end{aligned}$$

Finally, for the action we find

$$\Delta S = T_{M5}\int_\Sigma d^6x\sqrt{-G}\sum_{k=1}^{3}\Big\{\Delta\mathcal{L}^{(k)}_1 + \Delta\mathcal{L}^{(k)}_2 + \Delta\mathcal{L}^{(k)}_3 + \mathcal{O}(\Psi^2,\theta^5)\Big\} \qquad \text{(C.13)}$$

where

$$\begin{aligned}
\Delta\mathcal{L}^{(k)}_1 &= \frac{1}{2}\sqrt{det(A_r{}^s)}\sum_{i=1}^{k}\Big(\Big\{1+\frac{1}{2}tr(A^{-1}B)+\frac{1}{8}\Big(tr(A^{-1}B)\Big)^2-\frac{1}{4}tr(A^{-1}B)^2\Big\} \\
&\qquad\qquad \times(A^{-1}-A^{-1}BA^{-1}+A^{-1}BA^{-1}BA^{-1})^{mn}\Big)^{(k-i)}(\Delta g_{mn}-i\Delta\widetilde{H}_{mn})^{(i)} \\
A_{mn} &:= G_{mn}+i\widetilde{H}^{(0)}_{mn} \\
B_{mn} &:= (g_{mn}+i\widetilde{H}_{mn})^{(1)}+(g_{mn}+i\widetilde{H}_{mn})^{(2)}\ . \qquad \text{(C.14)}
\end{aligned}$$

Also

$$\begin{aligned}
\Delta\mathcal{L}^{(k)}_2 &= \frac{1}{4!\sqrt{-G}}\epsilon^{mnpqrs}\sum_{i=1}^{k}\Big\{\Big(H_{mnt}v_pv_lg^{tl}\Big)^{(k-i)}\Delta H^{(i)}_{qrs}+\Big(H_{qrs}v_pv_lg^{tl}\Big)^{(k-i)}\Delta H^{(i)}_{mnt} \\
&\quad + \Big(H_{qrs}H_{mnt}v_lg^{tl}\Big)^{(k-i)}\Delta v^{(i)}_p + \Big(H_{qrs}H_{mnt}v_pg^{tl}\Big)^{(k-i)}\Delta v^{(i)}_l \\
&\quad + \Big(H_{qrs}H_{mnt}v_pv_l\Big)^{(k-i)}\Delta g^{tl}_{(i)}\Big\} \\
\Delta\mathcal{L}^{(k)}_3 &= -\frac{1}{6!\sqrt{-G}}\epsilon^{mnpqrs}\Big(\Delta C^{(k)}_{mnpqrs}+10F_{mnp}\Delta H^{(k)}_{qrs}\Big)\,, \qquad \text{(C.15)}
\end{aligned}$$

where

$$\begin{aligned}
\Delta C^{(1)}_{m_1\ldots m_6} &= -6i(\Psi_{[m_1}\Gamma_{m_2\ldots m_6]}\theta) \\
\Delta C^{(2)}_{m_1\ldots m_6} &= 10(\Psi_{[m_1}\Gamma^{\underline{a}}\theta)(\mathcal{D}_{m_2}\theta\Gamma_{m_3\ldots m_6]\underline{a}}\theta)-5(\Psi_{[m_1}\Gamma_{m_2\ldots m_5|\underline{a}}\theta)(\mathcal{D}_{|m_6]}\theta\Gamma^{\underline{a}}\theta) \\
\Delta C^{(3)}_{m_1\ldots m_6} &= (\Psi_{[m_1}\mathfrak{G}\Gamma_{m_2\ldots m_6]}\theta)+(\Psi_{\underline{pq}}\mathcal{I}_{[m_1}{}^{\underline{pq}}\Gamma_{m_2\ldots m_6]}\theta)\ . \qquad \text{(C.16)}
\end{aligned}$$

The gravitino vertex operator at order θ^k is given by

$$V = T_{M5}\int_\Sigma d^6x\sqrt{-G}\Big(\Psi^{\underline{\alpha}}_{\underline{m}}V^{\underline{m}}_{\underline{\alpha}}+\Psi^{\underline{\alpha}}_{\underline{mn}}V^{\underline{mn}}_{\underline{\alpha}}\Big)\,, \qquad \text{(C.17)}$$

where

$$
\begin{aligned}
V_{\underline{\alpha}}^{\underline{m}} &:= \frac{\partial}{\partial \Psi_{\underline{m}}^{\underline{\alpha}}}\Big(\Delta\mathcal{L}_1^{(k)} + \Delta\mathcal{L}_2^{(k)} + \Delta\mathcal{L}_3^{(k)}\Big)|_{\Psi=0} \\
V_{\underline{\alpha}}^{\underline{mn}} &:= \frac{\partial}{\partial \Psi_{\underline{mn}}^{\underline{\alpha}}}\Big(\Delta\mathcal{L}_1^{(k)} + \Delta\mathcal{L}_2^{(k)} + \Delta\mathcal{L}_3^{(k)}\Big)|_{\Psi=0}
\end{aligned} \tag{C.18}
$$

and we have denoted by $\Psi_{\underline{mn}}^{\underline{\alpha}}$ the gravitino field strength: $\Psi_{\underline{mn}} := \mathcal{D}_{[\underline{m}}\Psi_{\underline{n}]}$. Substituting the preceding formulæ in (C.18) we arrive at the following explicit expressions

Order θ.

$$
\begin{aligned}
V^{(1)\underline{mn}}_{\underline{\alpha}} =&0 \\
V^{(1)\underline{m}}_{\underline{\alpha}} =& -i\sqrt{det(A_i{}^j)}(A^{-1})^{(mn)}(\Gamma_m\theta)_{\underline{\alpha}}\partial_n X^{\underline{m}} \\
&+ \frac{\epsilon^{lpqrs}{}_m}{6\sqrt{-G}}\sqrt{det(A_i{}^j)}(A^{-1})^{[mn]}\Big\{(\Gamma_n\theta)_{\underline{\alpha}}\partial_l X^{\underline{m}} + (\Gamma_l\theta)_{\underline{\alpha}}\partial_n X^{\underline{m}}\Big\}a_p(F_{qrs} - c_{qrs}) \\
&+ \frac{\epsilon^{klpqrs}}{12\sqrt{-G}}\sqrt{det(A_i{}^j)}(A^{-1})_{kl}a_p \\
&\quad\times\Big\{(F_{qrs} - c_{qrs})\Big[a^m a^n(\Gamma_m\theta)_{\underline{\alpha}}\partial_n X^{\underline{m}} + (\Gamma^m\theta)_{\underline{\alpha}}\partial_m X^{\underline{m}}\Big] + 3(\Gamma_{qr}\theta)_{\underline{\alpha}}\partial_s X^{\underline{m}}\Big\} \\
&+ \frac{i\epsilon^{klpqrs}}{12\sqrt{-G}}a_k(F_{lpq} - c_{lpq}) \\
&\quad\Big\{(F_{rst} - c_{rst})\Big[a^t a^n(\Gamma_n\theta)_{\underline{\alpha}} + \frac{1}{2}(\Gamma^t\theta)_{\underline{\alpha}}\Big] + \frac{1}{2}(\Gamma_{rs}\theta)_{\underline{\alpha}}\Big\}a^m\partial_m X^{\underline{m}} \\
&+ \frac{i\epsilon^{klpqrs}}{24\sqrt{-G}}a_k a^n(F_{lpq} - c_{lpq})(F_{rs}{}^t - c_{rs}{}^t)(\Gamma_n\theta)_{\underline{\alpha}}\partial_t X^{\underline{m}} \\
&+ \frac{i\epsilon^{klpqrs}}{5!\sqrt{-G}}\Big\{15a^t a_k(F_{lpt} - c_{lpt})(\Gamma_{qr}\theta)_{\underline{\alpha}} - 10a^t a_k(F_{lpq} - c_{lpq})(\Gamma_{rt}\theta)_{\underline{\alpha}} \\
&\qquad\qquad - 5F_{klp}(\Gamma_{qr}\theta)_{\underline{\alpha}} - (\Gamma_{klpqr}\theta)_{\underline{\alpha}}\Big\}\partial_s X^{\underline{m}}\ .
\end{aligned} \tag{C.19}
$$

Order $\theta^2\mathcal{D}\theta$.

$$
\begin{aligned}
V^{(2)\underline{mn}}_{\underline{\alpha}} &= 0 \\
V^{(2)\underline{m}}_{\underline{\alpha}} &= -\Big\{(\Gamma^{\underline{a}}\theta)_{\underline{\alpha}}(\mathcal{D}^m\theta\Gamma_{\underline{a}}\theta) - 2(\Gamma_n\theta)_{\underline{\alpha}}(\mathcal{D}^{(m}\theta\Gamma^{n)}\theta) \\
&\quad - \frac{1}{6}(\Gamma^m\Gamma^{n\underline{a}}\theta)_{\underline{\alpha}}(\mathcal{D}_n\theta\Gamma_{\underline{a}}\theta) + \frac{1}{6}(\Gamma^m\Gamma^{np}\theta)_{\underline{\alpha}}(\mathcal{D}_n\theta\Gamma_p\theta)\Big\}\partial_m X^{\underline{m}} + \dots\ ,
\end{aligned} \tag{C.20}
$$

where the ellipses stand for terms which drop out in the case of normal flux (see below). We have also dropped terms proportional to $\Gamma^m\mathcal{D}_m\theta$, which do not contain the zero mode.

Order θ^3.

$$
\begin{aligned}
V^{(3)\underline{mn}}_{\underline{\alpha}} &= \frac{i}{6}(\mathcal{I}_{\underline{p}}{}^{\underline{mn}})_{\underline{\alpha}}{}^{\beta}V^{(1)\underline{p}}_{\underline{\beta}} \\
V^{(3)\underline{m}}_{\underline{\alpha}} &= \frac{i}{6}\mathfrak{G}_{\underline{\alpha}}{}^{\beta}V^{(1)\underline{m}}_{\underline{\beta}}\ .
\end{aligned} \tag{C.21}
$$

Normal flux. In the case of normal flux, i.e. when the world-volume two-form tensor is flat ($F_{mnp}=0$) and the pullback of the three-form potential onto the fivebrane vanishes ($c_{mnp}=0$), the previous expressions simplify considerably. In particular we have,

$$\begin{aligned} iV^{(1)}{}_{\underline{\alpha}}^{\underline{m}} &= (\Gamma^m\theta)_{\underline{\alpha}}\partial_m X^{\underline{m}} + \frac{\epsilon^{klpqrs}}{5!\sqrt{-G}}(\Gamma_{klpqr}\theta)_{\underline{\alpha}}\partial_s X^{\underline{m}} \\ &= 2(\Gamma^m\theta)_{\underline{\alpha}}\partial_m X^{\underline{m}} \,. \end{aligned} \tag{C.22}$$

In deriving the second equality above we have taken into account that

$$\frac{\varepsilon^{m_1\dots m_p m_{p+1}\dots m_6}}{p!\sqrt{-G}}\Gamma_{m_{p+1}\dots m_6} = -(-1)^{p(p-1)/2}\Gamma^{m_1\dots m_p}P^+\,, \tag{C.23}$$

where the projector P^+ is defined in equation (4.22), and we have noted that after gauge-fixing the physical fermion modes satisfy $\theta = P^+\theta$. Moreover, the terms of the form $\Psi\theta^2\mathcal{D}\theta$ can be read off of equation (C.20). Taking (C.23), (C.21) into account and Wick-rotating, we finally arrive at equation (3.5).

D. Notation/conventions

For the convenience of the reader we give here an index of our main conventions and notation.

$\underline{M}=(\underline{m},\underline{\mu})$: target-space bosonic, fermionic curved indices

$\underline{A}=(\underline{a},\underline{\alpha})$: target-space bosonic, fermionic flat indices

$M=(m,\mu)$: world-volume bosonic, fermionic curved indices (from the beginning of the paper, up to and including section 4.2)

$m,n,p,\dots$: bosonic indices along X (from section 4.3 to the end of the paper)

$A=(a,\alpha)$: world-volume bosonic, fermionic flat indices

$Z^{\underline{M}}=(X^{\underline{m}},\theta^{\underline{\mu}})$: bosonic, fermionic superembedding coordinates

$e_{\underline{m}}{}^{\underline{a}}$: the $\theta=0$ component of $E_{\underline{m}}{}^{\underline{a}}$

$\Psi_{\underline{m}}{}^{\underline{\alpha}}$: the $\theta=0$ component of $E_{\underline{m}}{}^{\underline{\alpha}}$

$g_{mn} := E_m{}^{\underline{a}}E_n{}^{\underline{b}}\eta_{\underline{ab}}$

$G_{\underline{mn}} := e_{\underline{m}}{}^{\underline{a}}e_{\underline{n}}{}^{\underline{b}}\eta_{\underline{ab}}$

$(\omega_{\underline{m}})_{\underline{\alpha}}{}^{\underline{\beta}}$: the spin connection compatible with $G_{\underline{mn}}$

$c_{\underline{mnp}}$: the $\theta=0$ component of $C_{\underline{mnp}}$

$G_{\underline{mnpq}} := 4\partial_{[\underline{m}}c_{\underline{npq}]}$

$(\mathcal{D}_{\underline{m}})^{\underline{\alpha}}{}_{\underline{\beta}} := \delta^{\underline{\alpha}}{}_{\underline{\beta}}\partial_{\underline{m}} - \frac{1}{4}(\omega_{\underline{m}})^{\underline{\alpha}}{}_{\underline{\beta}} + \frac{1}{36}\Big((\Gamma^{\underline{abc}})^{\underline{\alpha}}{}_{\underline{\beta}}G_{\underline{mabc}} - \frac{1}{8}(\Gamma_{\underline{m}}{}^{\underline{abcd}})^{\underline{\alpha}}{}_{\underline{\beta}}G_{\underline{abcd}}\Big)$

$(\mathcal{R}_{\underline{mn}})^{\underline{\alpha}}{}_{\underline{\beta}}$: the curvature of the supercovariant derivative $(\mathcal{D}_{\underline{m}})^{\underline{\alpha}}{}_{\underline{\beta}}$

Convention: On x-space forms we convert between flat and curved indices using $e_{\underline{m}}{}^{\underline{a}}$.

Convention: We pull-back superforms (x-space forms) onto the world volume using $\partial_m Z^{\underline{M}}$ ($\partial_m X^{\underline{m}}$). Hence $e_m{}^{\underline{a}} := \partial_m X^{\underline{m}}e_{\underline{m}}{}^{\underline{a}}$, but $E_m{}^{\underline{a}} := \partial_m Z^{\underline{M}}E_{\underline{M}}{}^{\underline{a}}$.

Convention: We raise/lower curved world-volume indices on superforms using the Green-Schwarz metric g_{mn}

Convention: We raise/lower curved world-volume indices on x-space forms using the metric G_{mn}

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